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Reviewing two perfect fluids near a weak null singularity in spherical symmetry

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Abstract. This brief overview examines the contrasting behaviour of spherically symmetric dust versus spherically symmetric stiff fluid in a class of spherically symmetric black hole spacetimes which feature a weak null singularity (WNS). Under certain admissibility properties, we prove that the flow of the dust velocity does not experience any shell-crossing before or at the singularity, the dust velocity vector field remains timelike, and the dust energy density remains bounded as we approach the WNS. On the other hand, we show that the stiff fluid energy density becomes infinite as we approach the WNS and the stiff fluid velocity vector field approaches an ingoing null vector field tangent to the singular hypersurface.

1 Assumptions on the background spacetime

We work in a spherically symmetric, time-oriented 3+1 Lorentzian manifold $\mathcal{R} = \mathcal{Q}_{\mathcal{R}} \times S^2$; $g = -e^{2\omega}dudv + r^2 g_{S^2}$, covered by a global double null coordinate chart (u, v, θ, φ) where $\mathcal{Q}_{\mathcal{R}} = \{(u, v) : u < u_s, v > v_\kappa, uv > \text{const}\}$ is called the quotient manifold. For the readers familiar with scalar field perturbations of subextremal Reissner-Nördstrom, $(\mathcal{R}, g)$ may be viewed as a subset of the black hole interior of spacetimes studied in [1] and [2]. The lapse function e^ω and the area radius r are smooth and positive functions in (u, v), and extend continuously as positive functions to the hypersurface $\{v = 0\}$ which can be attached as a future boundary of $\mathcal{R}$, and constitutes a Cauchy horizon for the spacetime. Penrose diagram is provided in Figure 1. In addition, the future boundary component $\{v = 0\}$ is a *weak null singularity (WNS)*. More precisely, for the null derivatives of the area radius function we assume that $r_{,v} \to -\infty$ as $v \to 0^-$ for all $u \in (-\infty, 0)$ and that for each $u_0 < 0$ there exists a $v_0 = v_0(u_0) < 0$ so that $r_{,u}(u_0, v) < 0$ for all $v_0 < v \leq 0$. Roughly, '*the surface is singular*'. On the other hand, we assume the following bounds for metric functions and their derivatives in $\mathcal{R}$: $\exists C_g > 0$ depending only on the global geometry, such that for every $u_1 < u_2$ and $v_1 < v_2$ with $(u_i, v_j) \in \mathcal{Q}_{\mathcal{R}}$: $\left\|e^{\pm 2\omega}\right\|_{C(\mathcal{R})} + \left\|r^{-1}\right\|_{C(\mathcal{R})} + \|r_{,u}\|_{C(\mathcal{R})} \leq C_g$; $\|\omega_{,u}(\cdot, v_j)\|_{L^1[u_1,u_2]} + \|r_{,u}(\cdot, v_j)\|_{L^1[u_1,u_2]} \leq C_g$; $\|\omega_{,v}(u_i, \cdot)\|_{L^1[v_1,v_2]} + \|r_{,v}(u_i, \cdot)\|_{L^1[v_1,v_2]} \leq C_g$; $\|\omega_{,uv}(u_i, \cdot)\|_{L^1[v_1,v_2]} \leq C_g$. Roughly, the WNS is '*weak i.e. not too singular*'.

2 Main results

2.1 Dust

In the first part of [3] we investigate spherically symmetric perfect pressureless fluid (dust). The energy momentum tensor is $T_{ab} = \rho U_a U_b$. For the dust velocity U and energy density ρ, energy-momentum conservation implies that the geodesic equation $\nabla_U U = 0$ and the transport equation $\nabla_U(\rho) + \rho \nabla \cdot U = 0$ are satisfied. The fluid problem reduces to a Cauchy problem for a variation $\Gamma : \mathbb{R}^2 \supset D \to \mathcal{Q}_{\mathcal{R}}$ of infalling timelike geodesics which terminate on the WNS $\{v = 0\} = \mathcal{CH}^+$. This variation of geodesics determines the particle trajectories in the 3+1 dimensional spacetime $(\mathcal{R}, g)$. The initial data for this problem is supplied on an *admissible curve* λ (meaning λ has to satisfy certain geometric properties), which lifts to a spacelike hypersurface in $\mathcal{R}$.

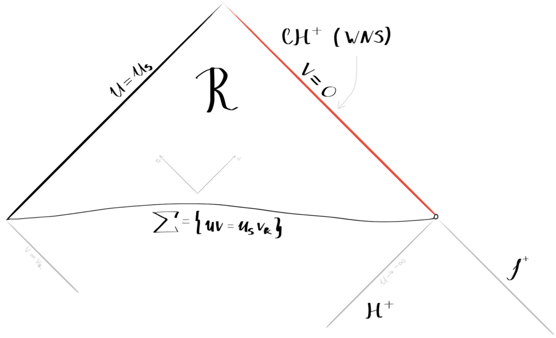

Figure 1: Penrose diagram of the background WNS spacetime $(\mathcal{R}, g)$.

As a consequence of the geodesic equation, we are able to show that the dust trajectories remain bounded away from null. By analysing the IVP for the Jacobi equation with initial data on the admissible curve λ, we prove that the Jacobi field is bounded and bounded away from zero along the geodesics emanating from λ. Hence we are able to establish that the variation of geodesics Γ based on an admissible curve λ is a diffeomorphism onto its image (no shell-crossing). Finally, we return to the fluid equations and apply the transport equation to obtain a bound for the energy density ρ of the dust in terms of the initial value of ρ on the hypersurface Image$(\lambda) \times S^2$, $\rho \leq C_\rho \|\rho\|_{C^0(\lambda)}$. The bounding constant depends on the global geometric properties of $(\mathcal{R}, g)$ (i.e. on C_g) and on the quantities characterising the admissibility of λ. The reason we can establish this bound is because the unbounded geometric quantities, namely $|\partial_v r|$ and $|\partial_v \omega|$, appear only inside integrals over outgoing null intervals. These L^1 norms are bounded in view of the spacetime assumptions. The result is illustrated in Figure 2.

2.2 Stiff fluid

In the second part of [3] we investigate a perfect fluid with pressure $p(\rho) = \rho$, energy-momentum tensor $T_{ab} = 2\rho U_a U_b + \rho g_{ab}$. Energy-momentum conservation implies that the fluid velocity is divergence-free: $\nabla \cdot (\sqrt{\rho} U) = 0$. Because of the spherical symmetry, we can show that the fluid is irrotational: $d(\sqrt{\rho} U) = 0$. Therefore, there exists a scalar field ψ such that $\sqrt{\rho} U^a = \nabla^a \psi$, satisfying the wave equation: $\Box_g \psi = 0$.

Our main objective is to prove estimates for the behaviour of the wavefunction ψ near the WNS, then translate the estimates for ψ into results for the fluid variables ρ, U. The analysis is done using a characteristic IVP for the wave equation $\Box_g \psi = 0$. Initial data $\mathring{\psi}$ is supplied on the bifurcate null hypersurface $\mathcal{C} = \mathcal{C}_+ \cup \overline{\mathcal{C}_-}$ such that $\nabla^a \mathring{\psi}$ is timelike on $\mathcal{C}$. Here $\mathcal{C}_+ = [u_0, u_1] \times \{v_0\}$ and $\overline{\mathcal{C}_-} = \{u_0\} \times [v_0, 0)$. Under the spacetime assumptions, we may choose $\mathcal{C}$ to be such that $\partial_u r$ and $\partial_v r$ are both negative in $D^+(\mathcal{C})$. We should note that the duality between the fluid and wave equation breaks down when the gradient of the wavefunction fails to be timelike. Using a bootstrap argument and monotonicity techniques similar to those applied in [4] and [5], we prove that this gradient remains timelike throughout the domain of dependence of the characteristic initial data. This is a nontrivial result and heavily reliant on the area radius being decreasing towards the future inside $D^+(\mathcal{C})$. Unlike the case for dust, where all 'bad' terms appear only under the integral sign over a v-interval, we find that in the stiff fluid case $|\partial_v \psi|$ is

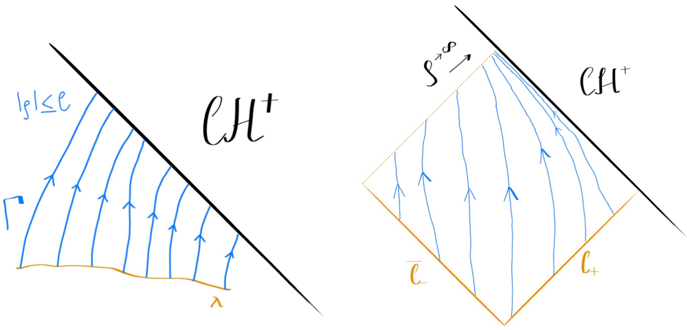

Figure 2: Illustrating the statements of the main theorems: left - dust; right - stiff fluid.

proportional to an integral of $|\partial_v r|$ over a u-interval. This integral blows at the WNS because of the assumption that $\partial_v r \to -\infty$ as $v \to 0^-$, leading to the blow up $\partial_v \psi \to -\infty$ as $v \to 0$. We also show that the u-derivative of ψ is bounded away from zero as $v \to 0$, for every u. Combining these, we prove that the energy density of the stiff fluid is unbounded at the WNS: $\rho = -2e^{-2\omega}\partial_u\psi\partial_v\psi \to \infty$ as $v \to 0$. Finally, showing that $\partial_v\psi/\partial_u\psi \to \infty$ at the WNS allows us to prove that the stiff fluid velocity field asymptotically approaches a null vector field near the WNS: $U^u = e^{-\omega}\sqrt{\psi_{,v}/\psi_{,u}} \to \infty$ and $U^v = e^{-\omega}\sqrt{\psi_{,u}/\psi_{,v}} \to 0$ as $v \to 0$. The stiff fluid blow up is illustrated in Figure 2.

2.3 Conclusion

Why can we prove opposing behaviour between the two fluids near the WNS under identical spacetime assumptions? While the dust has zero pressure, the stiff fluid has maximal pressure in a sense that it saturates the dominant energy condition. These physical differences lead to the mathematical differences outlined above, namely that for the dust case the unbounded geometric terms appear only inside integrals over outgoing null intervals, which are bounded by the spacetime assumptions, while for the stiff fluid unbounded geometric terms appear outside such integrals.

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