ON THE AXIOMATIC APPROACH TO QUANTUM FIELD THEORY

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In this paper a formulation of the fundamental axioms of the quantum field theory is proposed which is slightly modified from the usual formulation. The condition of minimum singularity of the commutator of the field and current operators for equal times is used as one of the axioms. Microcausality and the existence of a unitary S-matrix is a consequence of the initial axioms of the theory. These axioms make it possible to introduce a system of closed equations for the matrix elements of the S-matrix, which are extrapolated beyond the mass shell only with respect to one of the external 4-momenta. To eliminate undefined subtraction terms, the equations are written in an integral differential form in momentum space. When writing the equations in such a form it is possible to formulate the boundary conditions and show that the number of independent constants involved in the equations is exactly equal to the number of matrix elements which do not vanish when the invariant variables tend to infinity. The iterative solution of the obtained equations agrees with the renormalized series of the perturbation theory.

For simplicity a scalar field with a mass m without bound states is considered.

The fundamental axioms are (briefly):

1. The state space possesses a positive-definite metric;
2. A Hermitian operator $\phi(x)$ is associated with each point x in this space;
3. Lorentz-invariance;
4. Positive-definiteness of the energy spectrum;
5. The current operator $j(x) \equiv (\Box - m^2)\,\phi(x)$ commutes with $\phi(x)$ for equal times as follows:

$$[\varphi(x),\ j(y)]\,|_{x_0=y_0} = 0. \qquad (1)$$

For spinor operators the corresponding relation contains on the right $\delta(x-y)$.

6. The in-field operators

$$\varphi_{\rm in}(x) = \varphi(x) + \int \Delta^R(x-x',\ m)\, j(x')\, dx' \qquad (2)$$

satisfy the commutation relations

$$[\varphi_{\rm in}(x),\ \varphi_{\rm in}(y)] = -i\Delta(x-y,\ m).$$

and the space of in-states is complete.

The main difference from the commonly used formulations is contained in axiom 5. Usually in the axiomatic approach, the axiom of microcausality is introduced in order to define the local properties of the theory. It can be easily seen that microcausality in the sense of local commutativity, i.e.,

$$[j(x),\ \varphi(y)] = 0; \qquad (3)$$

$$(x-y)^2 < 0;$$

$$[j(x),\ j(y)] = 0, \qquad (4)$$

is a consequence of axioms 3, 5, and 6.

In [1] it is shown that from condition (3) and other axioms it follows that

$$[\varphi(x),\ \varphi(y)]_- = 0,\ (x-y)^2 < 0. \qquad (5)$$

Further it can be shown that from axioms 1 and 5 follows

$$\frac{\delta j(x)}{\delta\varphi_{\text{in}}(y)}-\frac{\delta j(y)}{\delta\varphi_{\text{in}}(x)}=-i\,[j(x),\ j(y)]. \qquad (6)$$

This is the existence criterion of a unitary S-matrix, expressed in terms of the properties of $j(x)$.

Let us find the integral differential equations which follow from the axioms. Let

$$r(1,\ldots,k\,|\,k+1,\ldots,m)\equiv \equiv\langle p_1,\ldots,p_k\,|\,j(0)\,|\,p_{k+1},\ldots,p_m\rangle. \qquad (7)$$

We have from relations (5) and (6)

$$r(1,\ldots,k\,|\,k+1,\ldots,m)= =R(1,\ldots,\tilde{i},\ldots,k\,|\,k+1,\ldots,m)+ +K(1,\ldots,0_i\ldots,k\,|\,k+1,\ldots,m), \qquad (8)$$

where

$$R=(2\pi)^{3/2}\sum_n\langle 1,\ldots,0_ik\,|\,j(0)\,|\,n\rangle\times \times\langle n\,|\,j(0)\,|\,k+1,\ldots,m\rangle\times \times\left\{\frac{\delta(\mathbf{p}_n-\mathbf{p}_k-\mathbf{p}_i)}{E_i+E_k-E_n-i\varepsilon}-\frac{\delta(\mathbf{p}_n-\mathbf{p}_n+\mathbf{p}_i)}{E_i+E_n-E_m-i\varepsilon}\right\};$$

and $K(1,\ldots,0_i,\ldots,k|k+1,\ldots,m)$ is an arbitrary function not depending on $\mathbf{p}_i$; 0_i denotes the absence of the vector $\mathbf{p}_i$.

Altogether in relations (8) there are m different expressions for the r-function $r(1,\ldots k|k+1,\ldots,m)$.

To eliminate the unknown K-terms from the the system (8), we make use of the invariant properties of R and K. We do this in the example of the 4-tail. Equation (8) in this case has the form

$$r(12/3)=\begin{cases} r(\tilde{1}\tilde{2}\,|\,3)+K(2\,|\,3),\\ R(1\tilde{2}\,|\,3)+K(1\,|\,3),\\ R(12\,|\,\tilde{3})+K(12\,|\,0),\end{cases} \qquad (9)$$

where $r(12/3)$ depends on the three invariants:

$$s=(p_1+p_2)^2;\ u=(p_2-p_3)^2;\ t=(p_1-p_3)^2.$$

Differentiating (9) with respect to the invariants on which the corresponding K-terms do not depend, we find the sought equations

$$\frac{\partial r}{\partial s}=\frac{\partial R(\tilde{1}2/3)}{\partial s};\ \frac{\partial r}{\partial u}=\frac{\partial R(1\tilde{2}/3)}{\partial u};\quad \frac{\partial r}{\partial t}=\frac{\partial R(12/\tilde{3})}{\partial t} \qquad (10)$$

and the additional conditions

$$\frac{\partial R(\tilde{1}2\,|\,3)}{\partial s}=\frac{\partial R(\tilde{1}2\,|\,3)}{\partial u};\ \frac{\partial R(\tilde{1}2\,|\,3)}{\partial u}=\frac{\partial R(12\,|\,\tilde{3})}{\partial u};$$

$$\frac{\partial R(\tilde{1}2\,|\,3)}{\partial t}=\frac{\partial R(12\,|\,\tilde{3})}{\partial t}. \qquad (11)$$

The elimination procedure of the K-terms is somewhat complicated starting from the 6-tail. For an arbitrary n-tail ($n=m+1$) the equation can be written in the form

$$\frac{\partial r(1,\ldots,k\,|\,k+1,\ldots,m)}{\partial s_i}=M_i(s_i,\ldots,s_N), \qquad (12)$$

where M_i is expressed in the form of linear combinations of the first derivatives of the R-functions; $s_1,\ldots,s_N$ are independent invariants, $N=3n-9$, $n>3$.

Besides this there are six additional conditions of a fairly cumbersome form.

We shall show that the additional conditions are a consequence of the exact equations for the r-functions. They are automatically satisfied when solving the equations with the help of perturbation theory (see [2], Appendix B).

If the system (4, 5) has a solution, it defines each r-function up to an arbitrary constant λ_n. A final solution by the perturbation theory is obtained only under the assumption that at infinity all the r-functions, except those of 3- and 4-tails, vanish. If this property is also conserved outside the framework of the perturbation theory, then the proposed system of axioms eliminates all nonrenormalized interaction variants.

In order to obtain the sought equation by the method of Lehmann, Symanzik, and Zimmermann [3], an additional assumption, equivalent to axiom 5, must be made.

We obtain expression (9) by assuming that the quasi-local terms which make a contribution to the mass shell [4] do not contain derivatives of δ-functions. Further elimination of quasilocal terms in no way differs from that described above.

Thus, the principal difference between the present work and other works on the axiomatic method consists in imposing additional limitations on the degree of singularity of the quasilocal terms and in their subsequent elimination from the equations.

A detailed account of the present work can can be found in [5] and in [2], where examples in perturbation theory are considered.

REFERENCES

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DISCUSSION

Yu.M. Shirokov

Does the rigorous fixing of the rate of growth serve as the only difference from the usual system of axioms?

V.Ya. Fainberg

Yes.

A.A. Logunov

You obtain actually the same system of equations as Medvedev and Polivanov?

V.Ya. Fainberg

Yes and no. The most important differences are such; firstly, limitations are imposed on the degree of singularity of the quasilocal terms (QLT) (in the language of the Bogolyubov-Medvedev-Polivanov method); secondly, in contrast to the works of Medvedev and Polivanov it is shown that for each amplitude (n-tail) there exists a system of $n-1$ equations with QLT and thirdly, on the basis of the covariant properties the QLT it is demonstrated that each amplitude not vanishing at infinity is defined by the equations up to an arbitrary constant.

M.K. Polivanov

Is it possible to say that your system of requirements selects minimal rates of growth which provide a nontrivial S-matrix in the theory?

V.Ya. Fainberg

In the case of scalar and pseudoscalar particles – yes. In the case of spin or particles, the system of axioms essentially agrees with that of Lehmann, Symanzik, and Zimmermann and, as before, is more rigorous than in the method of Bogolyubov, Medvedev, and Polivanov.