Classical and Quantum Symmetries in Models of Dimensionally Reduced Gravity

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Abstract

Dimensional reduction of various gravity and supergravity models leads to effectively two-dimensional field theories described by gravity coupled nonlinear $\mathbf{G}/\mathbf{H}$ coset space σ-models. This Thesis is devoted to an analysis of these models within the canonical framework, exploiting the close relations to well-known integrable field theories. A complete set of conserved nonlocal charges is derived from the transition and monodromy matrices of the associated linear system. Their Poisson algebra is a modified (twisted) version of the semi-classical Yangian double. The classical infinite-dimensional symmetry group (the Geroch group) is generated by the Lie-Poisson action of these charges. The structures completely extend to models with local supersymmetry, taking into account all additional fermionic degrees of freedom. Canonical quantization of the algebra of charges leads to a twisted Yangian double with fixed central extension at a critical level. The last chapter collects some results within the so-called isomonodromic approach to these models.

Zusammenfassung

Dimensionale Reduktion einer großen Klasse von Modellen höher-dimensionaler Gravitation und Supergravitation führt auf effektiv zwei-dimensionale Feldtheorien, genauer, auf gravitationsgekoppelte nichtlineare σ-Modelle auf Quotientenräumen $\mathbf{G}/\mathbf{H}$. Die vorliegende Arbeit ist einer Untersuchung dieser Modelle gewidmet. Dies geschieht im kanonischen Zugang, indem die engen Verbindungen zu bekannten integrablen Feldtheorien ausgenutzt werden. Ein vollständiger Satz erhaltener, nicht-lokaler Ladungen läßt sich aus den Monodromien des zugehörigen linearen Systems ableiten. Die Poisson-Algebra dieser Ladungen ist eine modifizierte (getwistete) Version des semi-klassischen Yangian-Doppels. Die unendlich-dimensionale klassische Symmetrie-Gruppe dieser Modelle (die Geroch Gruppe) wird durch die Lie-Poisson Wirkung der Ladungen erzeugt. Sämtliche Strukturen erweitern sich auf lokal supersymmetrische Modelle unter Berücksichtigung aller zusätzlichen fermionischen Freiheitsgrade. Die kanonische Quantisierung der Algebra nichtlokaler Ladungen führt auf ein getwistetes Yangian-Doppel mit zentraler Erweiterung. Das letzte Kapitel enthält eine Zusammenstellung von Resultaten im sogenannten isomonodromen Zugang zu diesen Modellen.

Contents

1 Introduction

The so-called hidden symmetries, appearing in the dimensional reduction of gravity and supergravity theories, have played an important role in the study of these theories over the last thirty years. Based on earlier work [32, 93] it was Geroch who first realized the emergence of an infinite-dimensional symmetry algebra in the two Killing vector field reduction of general relativity [47]. Later on, this symmetry structure was found to be generic for a broad class of models of dimensionally reduced gravity and supergravity theories [60, 62].

Upon reduction to two dimensions these models take the form of $\mathbf{G}/\mathbf{H}$ coset space σ-models coupled to $2d$ gravity and a dilaton. Various coset spaces descend from different models (see e.g. [67, 90, 62, 98, 14, 88, 45]), culminating in the $E_{8(+8)}/SO(16)$ which originates from dimensional reduction of maximally extended supergravity in eleven dimensions. The infinite-dimensional symmetry algebra of these models has been identified with the loop algebra which is associated with the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$; the existence of a central extension of this algebra has been noted in [61].

The interest in studying this class of two-dimensional models is (at least) a threefold. First, these models enlarge the list of integrable models, exhibiting a new underlying algebraic structure ((3.60), (3.61) below) which already deserves interest for itself: On the classical side we face a surprising regularization mechanism of the Poisson algebra of non-local charges – caused by the space-time coordinate dependence of the spectral parameter (3.3), which is one of the distinguished properties of the model. On the quantum side, the main interest is in the resulting algebra (5.5)–(5.9) below, which is a modification of the well-known Yangian double [28]. The twist by which it differs from the normal Yangian double essentially requires a new representation theory to be developed.

From the physical point of view, many of these models have received interest in the context of so-called midi-superspace models whose quantization serves as an interesting testing ground for many issues of quantum gravity. Despite the fact that dimensional reduction represents an essential truncation of the phase space, the models under consideration are sufficiently complicated to justify the hope that their exact quantization may provide insights into characteristic features of a still outstanding theory of quantum gravity. In particular, and in contrast to previously exactly quantized mini-superspace models, they exhibit an infinite number of degrees of freedom, which is broadly accepted to be a sine qua non for any significant model of quantum gravity. Their quantization may thus lead to progress in understanding the nature of quantum geometry and quantum black holes, reliability of semi-classical methods, etc. . This belief is e.g. supported by the observation that already rather simple and exactly soluble two-dimensional models of dilaton-coupled gravity capture and allow to further analyze several features that are expected to characterize quantum black hole solutions of the full four-dimensional theory (gravitational collapse, Hawking radiation, information loss, etc., see [116] and references therein).

Finally, from a higher-dimensional perspective these models and techniques find application in the study of gravitational string backgrounds and their symmetries, or describe the behavior of extended objects after dimensional reduction. It is further tempting to speculate about some higher-dimensional interpretation, where in a stringy setting the physical states of the theory, quantized on the two-dimensional world-sheet, are reinterpreted as the one-particle excitations of a higher-dimensional theory (see [102] for more speculation in this direction).

The interest in the symmetries of dimensionally reduced gravity originally arose in the context of the so-called solution-generating techniques [32, 78, 47, 67, 51]. Over the years, the point of view has changed. Rather than in producing new solutions to Einstein's field equations, nowadays, one is mainly interested in understanding the symmetry structures themselves. In particular, the analysis of the classical phase space with its full symmetry structure exhibited, is a necessary prerequisite for quantization. More precisely, a symmetry group which acts transitively on the phase space while preserving the symplectic structure may be identified with the classical phase space itself. The irreducible representations of this group then carry the information about the underlying quantum system.

The understanding of the structure of dimensionally reduced gravity was significantly improved by the revelation of the linear system [89, 7] which underlies the equations of motion. This established a first link to the integrable structures found in many two-dimensional models. It opened the possibility to subsequently make use of the methods and techniques which were developed in the theory of integrable systems (see [39] and references therein). In fact, the dimensionally reduced gravitational field equations (the Ernst equation [35] and its generalization to higher-dimensional Lie algebras) strongly resemble the equations of motion of the nonlinear σ-model [86, 121]; the main difference – apart from the coset structure – comes from the explicit appearance of the additional dilaton field in the gravitational equations. This field arises as a generic feature of Kaluza-Klein type dimensional reduction, measuring the size of the compactified (internal) manifolds. Throughout the following, it turns out to play a pivotal role.

For the nonlinear σ-models, it was soon realized that the arising (hidden) symmetries were not symplectic and generated by nonlocal charges which obeyed a new type of charge addition rules [86, 25], thus making manifest the nontrivial Hopf algebra structures of the underlying symmetry algebras. Since then, infinite-dimensional quantum groups have appeared to play a major role in lower-dimensional physics, providing a powerful description of the quantum symmetries of many integrable models and field theories. The classical symmetry generated by the nonlocal charges gains a natural description in the framework of Lie-Poisson actions [113, 6]. In particular, this offers new perspectives in quantization [11, 84] where the classical action turns into the adjoint representations of the underlying Hopf algebras.

Since it will become important in the following, let us mention a prominent example of the infinite-dimensional quantum groups, namely the Yangian algebra $Y(\mathfrak{g})$ associated with a simple finite-dimensional Lie algebra $\mathfrak{g}$. Having turned up already in the early days of the quantum inverse scattering method [114, 37], this algebra was rigorously defined within the framework of Hopf algebras by Drinfeld [27], and later on appeared to underlie many two-dimensional field theories (see [10, 12] and references therein). The Yangian algebra $Y(\mathfrak{g})$ may be considered as a deformation of the positive half of a loop algebra with nontrivial Hopf algebra structure. A deformation of the full loop algebra emerges from the Yangian double construction [28] which has been introduced in quantum field theory in [82, 11]. Like the loop algebra, this structure admits a central extension [110].

It is the purpose of this thesis to carry out the canonical framework for the described class of models of dimensionally reduced gravity by making use of the powerful tools that are provided by integrability and the emergence of quantum groups. The existence of a (modified) Yangian symmetry in the classical theory eventually allows the complete quantization. The results are essentially based on [72]–[77] and [104, 105].

The plan of the thesis is the following. In Chapter 2 we introduce the general class of two-dimensional coset space σ-models that shall play the main role in the text. The canonical formalism is set up, including the fundamental Poisson brackets and the gauge algebra of constraints. For illustration, we begin with a detailed discussion of the simplest model of the series – the two Killing vector field reduction of general relativity – and show how in this case the infinite-dimensional symmetry algebra arises.

Chapter 3 is devoted to the analysis of the classical integrability of the model. Starting from the linear system, we identify integrals of motion encoded in the associated transition and monodromy matrices. They are shown to be gauge invariant. We discuss, for which sectors of the theory this set of nonlocal charges is complete. This is essentially related to certain assumptions on the global behavior of the dilaton field. In the relevant sector (corresponding to a cylindrically symmetric setting) the nonlocal charges turn out to carry the values of the original physical fields on the symmetry axis. The Poisson algebra of these charges is computed. Again, the dilaton field plays a key role in that it causes the vanishing of certain ambiguities that are known to arise in the related structures in flat space σ-models. The resulting Poisson algebra is closely related to the Yangian double from which it differs by a twist which is remnant of the underlying coset structure. We end up with a reformulation of the classical model in terms of a complete set of nonlocal conserved charges. This formulation reveals integrability and the classical symmetry structure in a natural way. The Geroch group is recovered as the adjoint Lie-Poisson action associated with these nonlocal charges.

Chapter 4 contains the generalization of the structure to the maximally supersymmetric extension of the model, which gives rise to $N\!=\!16$ supergravity coupled to an $E_{8(+8)}/SO(16)$ coset space σ-model. Nonlocal charges may be defined in analogy to the bosonic case. Remarkably, they turn out to be supersymmetric, i.e. invariant under the full gauge superalgebra, and satisfy the same Poisson algebra as their purely bosonic counterparts. The essential calculations are performed in all fermionic orders, i.e. including all cubic fermionic terms that have been neglected so far.

In Chapter 5 we address the quantization of the model in terms of the nonlocal charges, i.e. search for the quantum algebra which reproduces the Poisson algebra in a classical limit while preserving certain extra properties (again related to the coset structure). We identify this algebra for the coset spaces $\mathbf{G}/\mathbf{H}\!=\!SL(N)/SO(N)$. The central result is given by the algebraic structure (5.5)–(5.9) below. In contrast to the well-known centrally extended Yangian double, the quantum R-matrices appear with a relative "twist" in the exchange relations which connect the two Yangian halves. A central extension of the algebra is required, whose value is uniquely fixed.

Finally, Chapter 6 contains several results obtained within the so-called isomonodromic framework, initiated in [71]. This approach has mainly been motivated by the apparent similarity of the equations of motion in certain sectors of the models under consideration with the deformation equations of monodromy preserving deformations [58]. Despite the rich mathematical structure which culminates in a link to the Knizhnik-Zamolodchikov equations from conformal field theory [68] (again slightly modified due to the underlying coset structure), we have so far not been able to embed this approach into the canonical framework which has been elaborated in the rest of the thesis.

In Chapter 7 we briefly summarize the solved and some remaining problems.

2 Models of Dimensionally Reduced Gravity

In this chapter, we introduce the class of models that we are going to study in the sequel. Originating from Kaluza-Klein type dimensional reduction of gravity and supergravity theories, they are casted into the form of two-dimensional $\mathbf{G}/\mathbf{H}$ coset space σ-models coupled to dilaton gravity. We discuss in detail the simplest example of this series, the two Killing vector field reduction of four-dimensional Einstein gravity, which is embedded into the general scheme with the particular coset space $\mathbf{G}/\mathbf{H} = SL(2,\mathbb{R})/SO(2)$. For this model, we give an elementary construction of the infinite dimensional symmetry algebra $\widehat{\mathfrak{sl}_2}$ due to Geroch [47]. In the next chapter, we will recover this symmetry within the general setting. Finally, we establish the general canonical formalism, including the Poisson brackets of the physical fields and the conformal gauge algebra.

2.1 The two Killing vector field reduction of Einstein gravity

The existence of two commuting Killing vector fields in four-dimensional general relativity gives rise to an essential simplification of the field equations and to a remaining model with a remarkably rich symmetry structure. In the following, we will describe this reduction and the arising of the symmetries.

Denote the four-dimensional metric by G_{MN} and consider the decomposition into the vierbein $E_M{}^A$

$$G_{MN} = E_M{}^A E_N{}^B \eta_{AB} \ , \tag{2.1}$$

with the Minkowski metric $\eta_{AB} = \mathrm{diag}(1,-1,-1,-1)$. Vacuum general relativity in four dimensions is described by the Lagrangian

$$\mathcal{L}^{(4)}_{\mathrm{EH}} \ = \ -\tfrac{1}{2} E^{(4)} R^{(4)} \ , \tag{2.2}$$

where $R^{(4)}$ and $E^{(4)}$ denote the curvature scalar of G_{MN} and the determinant of the vierbein E_M^A, respectively. The action is manifestly invariant under diffeomorphisms generated by vector fields ξ:

$$\delta_\xi E_M{}^A \ = \ \xi^N \partial_N E_M{}^A + E_N{}^A \partial_M \xi^N \ , \tag{2.3}$$

and Lorentz transformations generated by $\Lambda \in SO(1,3)$:

$$\delta_\Lambda E_M{}^A = E_M{}^B \Lambda_B{}^A \ . \tag{2.4}$$

Assume now the existence of two commuting Killing vector fields. For definiteness we take them to be spacelike, one of them with closed orbits. This characterizes spacetimes with cylindrical symmetry. It is convenient to adopt a coordinate system such that the Killing vector fields are given along coordinates $\frac{\partial}{\partial\phi}$ and $\frac{\partial}{\partial z}$, respectively. In this system, the coefficients of the metric depend only on the two remaining coordinates x and t. Further fixing the freedom of Lorentz transformations, the vierbein is casted into the block triangular form:

$$E_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & B_\mu^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix} . \tag{2.5}$$

Greek indices α, μ represent the coordinates x and t whereas small Roman indices a, m denote the coordinates ϕ and z associated to the Killing vector fields. We further parametrize the constituent $e_m{}^a$ of (2.5) by its determinant $\rho \equiv \det e_m{}^a$ and an $SL(2,\mathbb{R})$ matrix $\mathcal{V}$:

$$e_m{}^a \equiv \rho^{\frac{1}{2}} \mathcal{V} . \tag{2.6}$$

Inserting (2.5) into the original Lagrangian (2.2) leads after some calculation (see e.g. [13]) and up to surface terms to the following effectively two-dimensional Lagrangian

$$\begin{aligned} \mathcal{L}^{(2)} &= -\tfrac{1}{2}\rho E^{(2)} R^{(2)} + \tfrac{1}{8}\rho E^{(2)} h^{\mu\nu}\, \mathrm{tr}\left(\partial_\mu M M^{-1} \partial_\nu M M^{-1}\right) \\ &\qquad + \tfrac{1}{8}\rho E^{(2)} h^{\mu\kappa} h^{\nu\lambda} M_{mn} F^m_{\mu\nu} F^n_{\kappa\lambda} + \tfrac{1}{2} E^{(2)} h^{\mu\nu} \rho^{-1} \partial_\mu \rho \partial_\nu \rho , \end{aligned} \tag{2.7}$$

with

$$\begin{aligned} h_{\mu\nu} &\equiv e_\mu{}^\alpha e_\nu{}^\beta \eta_{\alpha\beta} \\ M_{mn} &\equiv (\mathcal{V}\mathcal{V}^T)_{mn} \;=\; \rho\, e_m{}^a e_n{}^b \delta_{ab} , \\ F^m_{\mu\nu} &\equiv \partial_\mu B^m_\nu - \partial_\nu B^m_\mu . \end{aligned}$$

The curvature scalar $R^{(2)}$ here corresponds to the two-dimensional metric $h_{\mu\nu}$; $E^{(2)}$ accordingly denotes the determinant of the zweibein $e_\mu{}^\alpha$.

From a lower dimensional point of view, the Lagrangian (2.7) describes two-dimensional gravity $h_{\mu\nu}$ coupled to scalar and vector matter fields which descend from the remaining components of the original higher-dimensional metric (2.5). The so-called Kaluza-Klein vector fields B^m_μ enter the Lagrangian only via their field strengths $F^m_{\mu\nu}$; they will prove to be auxiliary in the reduced theory. The matrix M combines the scalar fields which in two dimensions appear similar to the nonlinear σ-model coupled to gravity. They will play the main role in the sequel. The presence of the dilaton field ρ in (2.7) is a typical feature of Kaluza-Klein type dimensional reduction. In general context, this dilaton field measures the size of the compactified dimensions of the higher-dimensional space-time (cf. (2.5), (2.6)).

At least locally, the zweibein $e_\mu{}^\alpha$ may further be brought into diagonal form (conformal gauge) exploiting the freedom of the diffeomorphisms and Lorentz transformations in x^μ:

$$e_\mu{}^\alpha = \delta_\mu^{\ \alpha} \exp\sigma , \quad h_{\mu\nu} = \eta_{\mu\nu} \exp 2\sigma . \tag{2.8}$$

In the following, we neglect possible global obstructions. We introduce light-cone coordinates $x^\pm \equiv x^0 \pm x^1$ and similarly define $V^\pm \equiv V^0 \pm V^1$ and $V_\pm \equiv \frac{1}{2}(V_0 \pm V_1)$ for any vector V^μ and covector V_μ, respectively. The two-dimensional metric $h_{\mu\nu}$ then has components

$$h_{+-} = -\tfrac{1}{2} \exp 2\sigma . \tag{2.9}$$

In this model, it is not possible to gauge away the conformal factor σ since the Lagrangian (2.7) is not Weyl invariant, i.e. it is not invariant under local rescaling of the two-dimensional metric $h_{\mu\nu}$. The σ-model part of (2.7) is conformally coupled, but neither the coupling of the Kaluza-Klein vector fields nor the two-dimensional dilaton-gravity part is Weyl invariant. The reason for the latter is the multiplicative appearance of the dilaton field ρ, this is in contrast to usual $2d$ gravity.

Inherited symmetries of the lower-dimensional theory

Some of the gauge symmetries (2.3), (2.4) of the original theory are still compatible with the truncation (2.5), (2.8).

- Conformal transformations $\xi^{\pm}(x^{\pm})$ leave the form (2.8) invariant. According to (2.3) the fields transform as

$$\begin{aligned} \delta_{\xi^\pm}\mathcal{V} &= \xi^{\pm}\partial_{\pm}\mathcal{V} , \\ \delta_{\xi^\pm}\rho &= \xi^{\pm}\partial_{\pm}\rho , \\ \delta_{\xi^\pm}\sigma &= \xi^{\pm}\partial_{\pm}\sigma + \tfrac{1}{2}\partial_{\pm}\xi^{\pm} . \end{aligned} \tag{2.10}$$

- The special diffeomorphisms $\xi^m(x^\mu)$ act as gauge transformations on the Kaluza-Klein vector fields B_μ^m:

$$\delta_\xi B_\mu^m = \partial_\mu \xi^m . \tag{2.11}$$

- The linear diffeomorphisms $\xi^n = g_m{}^n x^m$ act as constant linear transformations on $\mathcal{V}$:

$$\delta_g \mathcal{V} = g\mathcal{V} , \qquad \text{with} \quad g = (g_m{}^n) \in SL(2,\mathbb{R}) . \tag{2.12}$$

  Upon toroidal compactification, i.e. with periodic boundary conditions on the directions x^m only a discrete subgroup $SL(2,\mathbb{Z})$ appears as gauge symmetry of the original theory. In any case however, (2.12) remains a symmetry of the lower-dimensional theory.

- The Lorentz transformations $\Lambda_a{}^b = h_a{}^b(x^\mu)$ act on $\mathcal{V}$ according to

$$\delta_h \mathcal{V} = \mathcal{V}h(x^\mu) , \qquad \text{with} \quad h(x^\mu) = (h_a{}^b)(x^\mu) \in SO(2) . \tag{2.13}$$

In abstract language, the physical degrees of freedom in $\mathcal{V}(x)$ parametrize the coset space $\mathbf{G}/\mathbf{H} = SL(2,\mathbb{R})/SO(2)$. The $\mathbf{H}$ gauge transformations are given by (2.13); the group $\mathbf{G}$ acts linearly by (2.12). One may choose a fixed system of representatives of the coset space, e.g. the triangular matrices $\mathcal{V}$.[1] The action (2.12) then provides a nonlinear realization of $SL(2,\mathbb{R})$:

$$\delta_g \mathcal{V} = g\mathcal{V} + \mathcal{V}h_g(x^\mu) , \tag{2.14}$$

[1] For general Lie groups one may correspondingly fix the orthogonal part of the Iwasawa decomposition of the matrix $\mathcal{V}$ [52].

where a compensating $SO(2)$ rotation h_g is required to restore triangularity of $\mathcal{V}$. This symmetry of the dimensionally reduced theory has been made explicit by Matzner and Misner [93]. Note that the matrix $M = \mathcal{V}\mathcal{V}^T$ is invariant under (2.13) and transforms linearly under (2.12).

Equations of motion

In conformal gauge (2.8) and after rescaling $\sigma \mapsto \sigma + \frac{1}{4}\ln\rho$ the Lagrangian (2.7) becomes (up to boundary terms again)

$$\mathcal{L}^{(2)} \;=\; -\partial_\mu\rho\,\partial^\mu\sigma + \tfrac{1}{8}\rho\Big(\mathrm{tr}\left(\partial_\mu M M^{-1}\partial^\mu M M^{-1}\right) + e^{-2\sigma}M_{mn}F^m_{\mu\nu}F^{n\,\mu\nu}\Big)\;, \qquad (2.15)$$

where the indices μ,ν are raised and lowered with the Minkowskian metric $\eta_{\mu\nu}$ now. The explicit appearance of the conformal factor σ shows, that (2.7) is not Weyl invariant. The equations of motion for the fields involved are the following:

- The Kaluza-Klein vector fields B^m_μ satisfy:

  $$\partial^\mu\left(e^{-2\sigma}\rho M_{mn}F^n_{\mu\nu}\right) = 0\;.$$

  In two dimensions this yields

  $$e^{-2\sigma}\rho M_{mn}F^n_{\mu\nu} = \mathrm{const}\;.$$

  In the following we restrict to that sector of the theory where the constant is zero. This is e.g. a necessary condition for asymptotically Minkowskian spacetimes.[2] The Kaluza-Klein vector fields then are (locally) pure gauge (2.11). They may carry physical degrees of freedom related to nontrivial topology of the two-dimensional surface parametrized by the x^μ. Neglecting these modes, in the following we restrict to the case

  $$B^m_\mu = 0\;. \qquad (2.16)$$

  The metric (2.1) then acquires block diagonal form, which is equivalent to hypersurface orthogonality of the Killing vectorfields: the surfaces orthogonal to both Killing vector fields are integrable.

- The dilaton field ρ obeys a free field equation:

  $$\Box\rho = 0\;. \qquad (2.17)$$

  Its general solution is given by $\rho(x) \equiv \rho^+(x^+) + \rho^-(x^-)$, and allows to introduce a dual field $\tilde\rho$

  $$\tilde\rho(x) \equiv \rho^+(x^+) - \rho^-(x^-)\;, \qquad (2.18)$$

[2] In addition, there are good arguments to believe that the rich symmetry structure of the model will not be compatible with nonvanishing cosmological constants of this type [100].

defined up to a constant. Under finite conformal gauge transformations (2.10), the field ρ transforms as

$$\rho \mapsto \rho^+(f^+(x^+)) + \rho^-(f^-(x^-)) \; , \tag{2.19}$$

with arbitrary functions f^+ and f^-. Assuming certain monotony behavior of ρ^+ and ρ^-, one may fix this residual gauge freedom by identifying the dilaton field with one of the two-dimensional world-sheet coordinates

$$\rho^+ = x^+ \; , \qquad \rho^- = \pm x^- \; . \tag{2.20}$$

The upper sign corresponds to a timelike dilaton field which appears e.g. in the context of the cosmological Gowdy models [49]. The lower sign refers to a spacelike dilaton field which has commonly been used in the description of gravitational waves with cylindrical symmetry [69, 79, 3]. With radial coordinate $\rho \equiv r$, the four-dimensional line element the takes the familiar form

$$ds^2 = e^{2\sigma(r,t)}(dt^2 - dr^2) - r\, M_{mn}(r,t) dx^m dx^n \; , \tag{2.21}$$

The distinguished coordinates (2.20) are often referred to as the Weyl canonical coordinates.

- The matter fields collected in the matrix $M = \mathcal{V}\mathcal{V}^T$ fulfill

  $$\partial_+ \left(\rho \partial_- M M^{-1}\right) + \partial_- \left(\rho \partial_+ M M^{-1}\right) = 0 \; . \tag{2.22}$$

  This is the so-called Ernst equation [35]. Except for the dilaton field ρ it agrees with the equations of motion of the nonlinear σ-model.

- The conformal factor σ satisfies two first order equations:

  $$\partial_\pm \rho \, \partial_\pm \hat{\sigma} = \tfrac{1}{2} \rho \, \mathrm{tr}\left(\partial_\pm M M^{-1} \partial_\pm M M^{-1}\right) \; , \tag{2.23}$$

  with $\hat{\sigma} \equiv \sigma - \frac{1}{2}\ln(\partial_+\rho\partial_-\rho)$. According to (2.10), $\hat{\sigma}$ transforms as a scalar under conformal transformations, making the conformal covariance of (2.23) manifest. Compatibility of these equations is ensured by (2.22). They determine the conformal factor up to a constant, since they are of first degree. Rather than equations of motion of the usual type, these equations form a set of (first-class) constraints. They are not derived from (2.15) but descend from variation of the two unimodular degrees of freedom of the $2d$ metric $h_{\mu\nu}$, that appear as Lagrangian multipliers in (2.7). The second order equation of motion for the conformal factor results from variation of the Lagrangian (2.15) w.r.t. ρ:

  $$\partial_+ \partial_- \hat{\sigma} = \partial_+ \partial_- \sigma = -\tfrac{1}{2} \mathrm{tr}\left(\partial_+ M M^{-1} \partial_- M M^{-1}\right) \tag{2.24}$$

  The consistency of this equation with the first order equations (2.23) can be checked using (2.17), (2.22) and (2.40).

The dual picture and the Geroch group

In addition to the gauge symmetries collected above, the two-dimensional model possesses a rich symmetry structure leading to complete integrability. This underlying structure becomes already manifest in a duality symmetry of the equations of motion, which we will describe in this subsection. In particular, this implies the existence of a dual of the (gauge) symmetry (2.12). Together with (2.12), it generates an infinite-dimensional symmetry group – the Geroch group.

In the next chapter, we will give a closed realization of this infinite-dimensional symmetry group and its action via the linear system and the associated transition matrices. Nevertheless, here we show how to generate the infinite-dimensional symmetry in an elementary way by successively commuting the two dual symmetry groups. Apart from giving a historical flavor, a construction of this type may turn out to be useful on the way to implement further symmetries in absence of a complete picture.[3]

The duality symmetry of this model appears as follows [13]. Parametrize the matrix $\mathcal{V}$ as

$$\mathcal{V} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho^{\frac{1}{2}}\Delta^{-\frac{1}{2}} & 0 \\ 0 & \rho^{-\frac{1}{2}}\Delta^{\frac{1}{2}} \end{pmatrix} , \tag{2.25}$$

where the gauge freedom (2.13) has been fixed to achieve triangularity. The equations of motion (2.22) then yield

$$\partial_+ \left(\Delta^2\rho^{-1}\partial_- B\right) + \partial_- \left(\Delta^2\rho^{-1}\partial_+ B\right) \;=\; 0 \,,$$

which gives rise to defining a dual potential B_D by

$$\partial_\pm B_D \equiv \pm\Delta^2\rho^{-1}\partial_\pm B \;. \tag{2.26}$$

With the further definition [78]

$$\mathcal{V}_D \equiv \begin{pmatrix} 1 & B_D \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Delta^{\frac{1}{2}} & 0 \\ 0 & \Delta^{-\frac{1}{2}} \end{pmatrix} , \tag{2.27}$$

it follows, that the matrix $\mathcal{V}_D$ satisfies the same equations of motion (2.22) with $M_D = \mathcal{V}_D\mathcal{V}_D^T$.

This duality has two interesting consequences. First, note the different asymptotic behavior of $\mathcal{V}$ and $\mathcal{V}_D$ at $\rho \to \infty$. E.g. in Weyl coordinates (2.21), $4d$-Minkowski space is described by $\Delta = 1$, $B = 0$. Thus, at radial infinity $\rho \to \infty$ the matrices $\mathcal{V}$ and $\mathcal{V}_D$ behave as

$$\mathcal{V} \to \begin{pmatrix} \rho^{-\frac{1}{2}} & 0 \\ 0 & \rho^{\frac{1}{2}} \end{pmatrix} , \quad \mathcal{V}_D \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \tag{2.28}$$

for asymptotically Minkowskian spacetimes. In a similar way, $\mathcal{V}$ and $\mathcal{V}_D$ differ on the symmetry axis $\rho = 0$. We can hence describe the same physical situation by equivalent models with different asymptotics.

[3] Since the Geroch group appears to be already transitive in the sector which we have described so far, additional symmetries can only enter when one restores more physical degrees of freedom. A promising candidate are e.g. the topological degrees of freedom of the Kaluza-Klein vector fields B_μ^m and of the two-dimensional metric $h_{\mu\nu}$, relaxing (2.16) and (2.8), respectively. Their relevance in the further reduction to one dimension has already been suggested in [100, 96].

Second and more important, since $\mathcal{V}_D$ obeys the same equations of motion (2.22), there is a dual symmetry to (2.14), which we denote by $SL(2,\mathbb{R})_D$. Via (2.26) the action of $SL(2,\mathbb{R})_D$ on the original fields $\mathcal{V}$ can be constructed and turns out to be rather nontrivial. This symmetry has originally been discovered by Ehlers [32] in the three-dimensional reduction of $4d$-Einstein gravity. The most interesting property of the two symmetry groups $SL(2,\mathbb{R})$ and $SL(2,\mathbb{R})_D$ is that they do not commute but span an infinite-dimensional symmetry group – the so-called Geroch group [47]. On the algebra level, $\mathfrak{sl}(2)$ and $\mathfrak{sl}(2)_D$ span the affine algebra $\widehat{\mathfrak{sl}_2}$.

Let us make this more explicit. Denote the generators of $\mathfrak{sl}(2)$ by h, e, f. According to (2.14) they act on $\mathcal{V}$ by left multiplication with the matrices

$$\mathbf{h} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad \mathbf{e} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \quad \mathbf{f} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \tag{2.29}$$

and a compensating $\mathfrak{so}(2)$-rotation induced by f. We now turn to the action of $\mathfrak{sl}(2)_D$ with generators h_D, e_D, f_D. Similarly to (2.14) they act on $\mathcal{V}_D$ as:

$$\begin{aligned} \delta_{h_D}\mathcal{V}_D &= \mathbf{h}\mathcal{V}_D = \begin{pmatrix} \Delta^{\frac12} & B_D\Delta^{-\frac12} \\ 0 & -\Delta^{-\frac12} \end{pmatrix} , \quad \delta_{e_D}\mathcal{V}_D = \mathbf{e}\mathcal{V}_D = \begin{pmatrix} 0 & \Delta^{-\frac12} \\ 0 & 0 \end{pmatrix} , \\ \delta_{f_D}\mathcal{V}_D &= \mathbf{f}\mathcal{V}_D - \mathcal{V}_D \begin{pmatrix} 0 & -\Delta \\ \Delta & 0 \end{pmatrix} = \begin{pmatrix} -B_D\Delta^{\frac12} & \Delta^{\frac32} \\ 0 & B_D\Delta^{-\frac12} \end{pmatrix} . \end{aligned} \tag{2.30}$$

Via (2.26), (2.27), this gives the action on $\mathcal{V}$:

$$\begin{aligned} \delta_{h_D}\mathcal{V} &= \begin{pmatrix} -\rho^{\frac12}\Delta^{-\frac12} & -\rho^{-\frac12}\Delta^{\frac12}B \\ 0 & \rho^{-\frac12}\Delta^{\frac12} \end{pmatrix} = -\delta_h\mathcal{V} , \quad \delta_{e_D}\mathcal{V} = 0 , \\ \delta_{f_D}\mathcal{V} &= \begin{pmatrix} \rho^{\frac12}B_D\Delta^{-\frac12} & \rho^{-\frac12}\Delta^{\frac12}\phi_D \\ 0 & -\rho^{-\frac12}B_D\Delta^{\frac12} \end{pmatrix} , \end{aligned} \tag{2.31}$$

with ϕ_D defined by

$$\partial_\pm\phi_D \equiv \pm\left(\rho\Delta^{-1}\partial_\pm\Delta - \rho^{-1}\Delta^2 B\partial_\pm B\right) . \tag{2.32}$$

Compatibility of these equations is again ensured by (2.22).

The algebraic structure of the symmetries becomes more transparent in their action on the currents $J_\pm \equiv \mathcal{V}^{-1}\partial_\pm\mathcal{V}$. These are left invariant by $\mathfrak{sl}(2)$ and transform only under f_D according to

$$\begin{aligned} \delta_{f_D}J_\pm &= \pm\begin{pmatrix} \rho^{-1}\Delta^2\partial_\pm B & 2\partial_\pm\Delta \\ 0 & -\rho^{-1}\Delta^2\partial_\pm B \end{pmatrix} \\ &= \pm\left[\mathcal{V}^{-1}\mathbf{e}\mathcal{V} , J_\pm + J_\pm^T\right] \pm 2\partial_\pm\rho\,\mathcal{V}^{-1}\mathbf{e}\mathcal{V} . \end{aligned} \tag{2.33}$$

This immediately gives rise to the next commutators (note that δ_f annihilates $J_\pm$ but not $\mathcal{V}$):

$$\begin{aligned} [\delta_f, \delta_{f_D}]J_\pm &= \pm\left[\rho\mathcal{V}^{-1}\mathbf{h}\mathcal{V} , J_\pm + J_\pm^T\right] \pm 2\partial_\pm\rho\,\mathcal{V}^{-1}\mathbf{h}\mathcal{V} , \\ [\delta_f, [\delta_f, \delta_{f_D}]]J_\pm &= \mp 2\left[\mathcal{V}^{-1}\mathbf{f}\mathcal{V} , J_\pm + J_\pm^T\right] \mp 4\partial_\pm\rho\,\mathcal{V}^{-1}\mathbf{f}\mathcal{V} . \end{aligned} \tag{2.34}$$

Upon further commuting, these transformations generate the affine algebra $\widehat{\mathfrak{sl}_2}$. As a vector space this algebra is given by $\mathfrak{sl}_2 \otimes C[z, z^{-1}] \oplus k\mathbb{C}$, where $C[z, z^{-1}]$ denotes the set of Laurent polynomials in a formal variable z. The algebraic structure is:

$$\begin{aligned} [h \otimes z^m, e \otimes z^n] &= 2e \otimes z^{m+n} , \quad [h \otimes z^m, f \otimes z^n] = -2f \otimes z^{m+n} , \\ [e \otimes z^m, f \otimes z^n] &= h \otimes z^{m+n} + k\delta^{m+n,0} . \end{aligned} \tag{2.35}$$

The element k lies in the center of $\widehat{\mathfrak{sl}_2}$ and is referred to as the central extension. The subalgebras $\mathfrak{sl}(2)$ and $\mathfrak{sl}(2)_D$ are embedded into $\widehat{\mathfrak{sl}_2}$ as follows:

$$\begin{aligned} & h = h \otimes z^0 , \quad e = e \otimes z^0 , \quad f = f \otimes z^0 , \\ & h_D = \tau(h) \otimes z^0 + k , \quad e_D = \tau(e) \otimes z^{-1} , \quad f_D = \tau(f) \otimes z , \end{aligned} \tag{2.36}$$

where τ is the algebra-involution $(h \mapsto -h, e \mapsto -f, f \mapsto -e)$. These two subalgebras correspond to the two nodes of the associated Dynkin diagram [64]. Together they obviously span the full algebra (2.35). The transformations from (2.33), (2.34) correspond to the elements $\mathfrak{sl}(2) \otimes z$.

We close this section with a few remarks on properties of the Geroch group, which have already shown up here.

Remark 2.1 The action of $\mathfrak{sl}(2)_D$ on $\mathcal{V}$ in (2.31) involves two dual potentials B_D (2.26) and ϕ_D (2.32) whose existence follows from the Ernst equation (2.22). By further commuting the transformations from $\mathfrak{sl}(2)$ and $\mathfrak{sl}(2)_D$ an infinite hierarchy of such dual potentials arises. They have been observed already in the early history of the Geroch group [47, 67]. On the level of associated charges, the construction of this hierarchy corresponds to the well-known procedure [15] of successively generating nonlocal charges in two-dimensional integrable models.

Remark 2.2 Equations (2.31) illustrate another property of the Geroch group. It is only the half $\mathfrak{sl}_2 \otimes C[z]$ of the affine algebra (2.35) which acts nontrivially on the physical fields. The other half $\mathfrak{sl}_2 \otimes C[z^{-1}]$ describes the freedom of shifting the dual potentials (c.f. the action of e_D in (2.30)). Accordingly, the central extension k in (2.35) leaves $\mathcal{V}$ invariant. However, it has been observed by Julia [61] that this central extension acts nontrivially on the conformal factor σ which is determined by $\mathcal{V}$ only up to a constant (2.23).

Remark 2.3 To honestly prove the existence of the affine symmetry (2.35) at this stage, one would have to check the corresponding Serre relations between multi-commutators of the generators (2.36) [100] as well as the absence of further relations between them. We refrain here from doing so since later on we will present a closed approach which makes the affine symmetry explicit.

2.2 Two-dimensional coset space σ-models coupled to gravity and a dilaton

Dimensionally reduced pure Einstein gravity described in the previous section already captures all the features of the class of models we are going to study. It is the simplest example of the $\mathbf{G}/\mathbf{H}$ coset space σ-models that arise from dimensional reduction of various gravity and supergravity models. More general, d-dimensional Einstein gravity with $(d-2)$ commuting Killing vector fields [90] gives rise to a $SL(d{-}2,\mathbb{R})/SO(d{-}2)$ coset space σ-model. Other examples with higher-dimensional coset spaces $\mathbf{G}/\mathbf{H}$ come from Einstein-Maxwell systems [67] and Einstein-Maxwell-dilaton-axion systems [45]. The largest exceptional – and maybe most fundamental – coset space $E_{8(+8)}/SO(16)$ arises from dimensional reduction of maximally extended $N=8$ supergravity in 4 dimensions [60, 62, 98]. For general reasons, related to boundedness of the energy, it is always the maximal compact subgroup $\mathbf{H}$ of $\mathbf{G}$ that is divided out in the coset.

Let Σ be a two-dimensional Lorentzian world-sheet, parametrized by coordinates x^μ. Let $\mathbf{G}$ be a semisimple Lie group and $\mathfrak{g}$ the corresponding Lie algebra with basis $\{t_A\}$. The Cartan-Killing form in the fundamental representation is given by $\mathrm{tr}(t_A t_B)$ and used to raise and lower algebra indices. Denote by $\mathbf{H}$ the maximal compact subgroup of $\mathbf{G}$, characterized as the fixgroup of an involution τ [52]. Lifting τ to the algebra gives rise to the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \qquad \text{with} \quad \tau(\xi) = \begin{cases} \xi & \text{for } \xi \in \mathfrak{h} \\ -\xi & \text{for } \xi \in \mathfrak{k} \end{cases} , \tag{2.37}$$

which is orthogonal with respect to the Cartan-Killing form. For instance, for the coset space $\mathbf{G}/\mathbf{H} = SL(N,\mathbb{R})/SO(N)$, the involution τ is defined by $\tau(X) \equiv (X^T)^{-1}$ for $X \in \mathbf{G}$ and $\tau(\xi) \equiv -\xi^T$ for $\xi \in \mathfrak{g}$, respectively.

The physical fields of the model are mappings $\mathcal{V}(x^\mu)$ from Σ into the coset space $\mathbf{G}/\mathbf{H}$, i.e. they are $\mathbf{G}$-valued and exhibit the gauge freedom of right $\mathbf{H}$-multiplication (cf. (2.13))

$$\mathcal{V} \;\mapsto\; \mathcal{V} H \; . \tag{2.38}$$

The currents $\mathcal{V}^{-1}\partial_\mu \mathcal{V}$ allow decomposition according to (2.37):

$$J_\mu \equiv J_\mu^A t_A \equiv \mathcal{V}^{-1}\partial_\mu \mathcal{V} \equiv Q_\mu + P_\mu \; ; \qquad \text{with} \quad Q_\mu \in \mathfrak{h} \, , \;\; P_\mu \in \mathfrak{k} \; . \tag{2.39}$$

These currents are subject to the compatibility relations

$$\begin{aligned} \partial_\mu Q_\nu - \partial_\nu Q_\mu + [Q_\mu, Q_\nu] + [P_\mu, P_\nu] &= 0 \, , \\ D_\mu P_\nu - D_\nu P_\mu &= 0 \, , \end{aligned} \tag{2.40}$$

with the ($\mathbf{H}$-)covariant derivative $D_\mu P_\nu \equiv \partial_\mu P_\nu + [Q_\mu, P_\nu]$. Under the gauge transformations (2.38) they transform as

$$Q_\mu \;\mapsto\; H^{-1} Q_\mu H + H^{-1}\partial_\mu H \; , \quad P_\mu \;\mapsto\; H^{-1} P_\mu H \; , \tag{2.41}$$

with $H = H(x^\mu) \in \mathbf{H}$. The matrix

$$M \equiv \mathcal{V}\, \tau(\mathcal{V})^{-1} \; , \tag{2.42}$$

is the analogue of the matrix containing the higher dimensional metric coefficients in (2.21). It is symmetric under

$$M = \tau(M)^{-1} \; , \tag{2.43}$$

and its current is related to the coset currents from (2.39) by

$$\partial_\mu M M^{-1} = 2\mathcal{V} P_\mu \mathcal{V}^{-1} \equiv 2 D_\mu \mathcal{V}\mathcal{V}^{-1} \; . \tag{2.44}$$

It is the separate task of each dimensional reduction to two dimensions to eventually cast the resulting model into the form of the corresponding coset space σ-model. In the last section this has been shown in detail for pure Einstein gravity with two commuting Killing vector fields. See [67, 90, 62, 98, 14, 45] for more complicated examples.

The final form of the two-dimensional Lagrangians and the corresponding equations of motion are a straight-forward generalization of (2.15)–(2.24) inserting the matrix $M \in \mathbf{G}$ from (2.42). The coset-structure becomes more transparent if we rewrite the currents in terms of the coset currents from (2.39): $\partial_\mu M M^{-1} = 2\mathcal{V} P_\mu \mathcal{V}^{-1} = 2 D_\mu \mathcal{V}\mathcal{V}^{-1}$. Summarizing, we obtain the Lagrangian

$$\mathcal{L}^{(2)} \;=\; -\partial_\mu \rho \, \partial^\mu \sigma + \tfrac{1}{2}\rho \, \mathrm{tr}\,(P_\mu P^\mu) \;\; , \tag{2.45}$$

and the equations of motion for

the dilaton field:

$$\Box \rho \;=\; 0 \; , \tag{2.46}$$

the conformal factor:

$$\partial_\pm \rho \partial_\pm \hat{\sigma} \;=\; \partial_\pm \rho \partial_\pm \sigma - \tfrac{1}{2}\partial_\pm \partial_\pm \rho \;=\; \tfrac{1}{2}\rho \, \mathrm{tr}\,(P_\pm P_\pm) \;\; , \tag{2.47}$$

$$\partial_+ \partial_- \hat{\sigma} \;=\; -\tfrac{1}{2}\mathrm{tr}\,(P_+ P_-) \;\; , \tag{2.48}$$

and the scalars building the coset space:

$$D_\mu(\rho P^\mu) \;=\; D_+(\rho P_-) + D_-(\rho P_+) \;=\; 0 \; . \tag{2.49}$$

The discussion accompanying these equations in (2.17)–(2.24) can be adopted for the general case here.

Remark 2.4 The Lagrangian (2.45) and the equations of motion for the currents $P_\pm$ resemble the principal chiral field model (PCM) [86, 38] with the compact group $\mathbf{G}$ of the PCM replaced by the noncompact coset manifold $\mathbf{G}/\mathbf{H}$ and arising of the additional dilaton field ρ. It is mainly the appearance of ρ that accounts for the new features of these models in comparison with the flat space models. Equations (2.47) further show that ρ may not be chosen constant without trivializing the matter part of the solution [99]. Since the Cartan-Killing form $\mathrm{tr}(t_A t_B)$ is positive definite on the coset $\mathfrak{k}$, $\partial_\pm \rho = 0$ would require $P_\pm = 0$. It is also seen from (2.10) that any solution with $\partial_\pm \rho = 0$ has some degenerate orbit under the conformal gauge transformations. There is hence no smooth limit in which the dilaton-coupled model would approach the PCM.

2.3 Canonical formalism

Poisson structure

In this paragraph, we derive the canonical Poisson structure from the Lagrangian (2.45). For simplicity, we denote the spatial coordinate x^1 by x only and the timelike coordinate x^0 by t.

Moreover, we drop the argument t in most of the following equations, keeping in mind, that the Poisson brackets are defined at equal times.

For the conformal factor σ and the dilaton field ρ we directly obtain:

$$\{\rho(x), \partial_0\sigma(y)\} = \{\sigma(x), \partial_0\rho(y)\} = -\delta(x-y) \;, \tag{2.50}$$

i.e. the conjugate momenta to ρ and σ coincide with $\partial_0\sigma$ and $\partial_0\rho$, respectively. These relations are equivalent to

$$\{\partial_\pm\rho(x), \partial_\pm\sigma(y)\} = \mp\tfrac{1}{2}\delta'(x-y) \;, \qquad \{\partial_\pm\rho(x), \partial_\mp\sigma(y)\} = 0 \;.$$

In terms of the fields ρ and $\tilde\rho$ from (2.18) the brackets (2.50) become

$$\{\rho(x), \partial_0\sigma(y)\} = \{\tilde\rho(x), \partial_1\sigma(y)\} = -\delta(x-y) \;. \tag{2.51}$$

There are also different ways to choose the canonical coordinates among the matrix entries of M. One may e.g. parametrize the matrix M by coordinates like in (2.42) which take into account the group properties and the additional symmetry (2.43) to then extract canonical brackets from (2.45). For higher dimensional groups $\mathbf{G}$ however, such a set of explicit coordinates is hard to find and certainly not very practicable. The algebra valued currents $\partial_\pm MM^{-1}$ offer a suitable parametrization but hide the symmetry property (2.43).

It is thus most convenient to consider the currents (2.39) of the matrices $\mathcal{V}$ as basic variables. Definition (2.42) then ensures (2.43). Moreover, the choice of $\mathcal{V}$ as fundamental objects is indispensable for coupling fermions to the model (cf. Chapter 4). The prize for introducing the additional $\mathbf{H}$-gauge freedom (2.38) in $\mathcal{V}$ is the appearance of the associated constraints (2.55) below.

In a standard way [39], we obtain the canonical Poisson structure with coordinates J_1. Introduce the corresponding momenta

$$\pi \equiv \pi_Q + \pi_P \equiv \frac{\delta S}{\delta J_1} \equiv \frac{\delta S}{\delta(\partial_0 J_1^A)}\, t^A \;, \tag{2.52}$$

with

$$\left\{J_1^A(x), \pi_B(y)\right\} = \delta^A_B\, \delta(x-y) \;, \tag{2.53}$$

at equal times. The time derivative of J_1 is expressed in terms of Q_0 and P_0 via the relations (2.40):

$$\partial_0(Q_1 + P_1) = \partial_0 J_1 = \partial_1 J_0 + [J_1, J_0] \equiv \nabla_1 J_0 \;.$$

The operator ∇_1 is linear and antisymmetric with respect to the scalar product $\left(\mathrm{tr}\int dx\right)$. The relevant part of the action (2.45) thus reads

$$\begin{aligned} \tfrac{1}{2}\int dx\, \rho\, \mathrm{tr}\,(P_0 P_0) &= \tfrac{1}{2}\int dx\, \rho\, \mathrm{tr}\left(P_0 \nabla_1^{-1}(\partial_0 J_1)\right) \\ &= -\tfrac{1}{2}\int dx\, \mathrm{tr}\left((\partial_0 J_1)\nabla_1^{-1}(\rho\, P_0)\right) \;, \end{aligned}$$

leading to

$$\rho\, P_0 = -\nabla_1 \pi = -\partial_1\pi - [J_1, \pi] \;.$$

Splitting this expression according to (2.37) implies

$$\begin{aligned} \rho\, P_0 &= -\partial_1 \pi_P - [Q_1, \pi_P] - [P_1, \pi_Q]\;, \\ 0 &= -\partial_1 \pi_Q - [Q_1, \pi_Q] - [P_1, \pi_P]\;. \end{aligned} \tag{2.54}$$

The second equation defines a set of weakly vanishing constraints

$$\Phi \;\equiv\; \Phi_A t^A \;\equiv\; \partial_1 \pi_Q + [Q_1, \pi_Q] + [P_1, \pi_P] \;\approx\; 0\;, \tag{2.55}$$

related to the gauge transformations (2.41).

Many calculations in the following are more conveniently performed in the index-free tensor notation. Denote for some matrix A^{ab}:

$$\overset{1}{A} \equiv A \otimes I \qquad \text{and} \qquad \overset{2}{A} \equiv I \otimes A\;.$$

In components this takes the form $(A \otimes I)^{ab,cd} \equiv A^{ab}\delta^{cd}$ and $(I \otimes A)^{ab,cd} \equiv A^{cd}\delta^{ab}$. Define accordingly the following matrix notation of Poisson brackets [39]:

$$\left\{ \overset{1}{A}\,,\, \overset{2}{B} \right\}^{ab,cd} \;\equiv\; \left\{ A^{ab}, B^{cd} \right\}\;, \tag{2.56}$$

for matrices A^{ab}, B^{cd}. Let $\Omega_{\mathfrak{g}} \equiv t_A \otimes t^A$ be the Casimir element of $\mathfrak{g}$, which due to orthogonality of the decomposition (2.37) allows the splitting $\Omega_{\mathfrak{g}} = \Omega_{\mathfrak{h}} + \Omega_{\mathfrak{k}}$. The canonical brackets (2.53) in this notation become

$$\left\{ \overset{1}{Q}_1(x)\,,\, \overset{2}{\pi}_Q(y) \right\} = \Omega_{\mathfrak{h}}\, \delta(x-y)\;, \qquad \left\{ \overset{1}{P}_1(x)\,,\, \overset{2}{\pi}_P(y) \right\} = \Omega_{\mathfrak{k}}\, \delta(x-y)\;.$$

Equation (2.54) now yields the Poisson brackets for the physical fields:

$$\begin{aligned} \left\{ \rho(x)\, \overset{1}{P}_0(x)\,,\, \overset{2}{\mathcal{V}}(y) \right\} &= -\, \overset{2}{\mathcal{V}}(x)\, \Omega_{\mathfrak{k}}\, \delta(x-y)\;, \\ \left\{ \rho(x)\, \overset{1}{P}_0(x)\,,\, \overset{2}{Q}_1(y) \right\} &= \left[\Omega_{\mathfrak{k}}\,,\, \overset{2}{P}_1(x) \right] \delta(x-y)\;, \\ \left\{ \rho(x)\, \overset{1}{P}_0(x)\,,\, \overset{2}{P}_1(y) \right\} &= \left[\Omega_{\mathfrak{k}}\,,\, \overset{2}{Q}_1(x) \right] \delta(x-y) + \Omega_{\mathfrak{k}}\, \partial_x \delta(x-y)\;, \\ \left\{ \rho(x)\, \overset{1}{P}_0(x)\,,\, \overset{2}{P}_0(y) \right\} &= \left[\Omega_{\mathfrak{k}}\,,\, \overset{2}{\Phi}(x) \right] \delta(x-y) \;\approx\; 0\;. \end{aligned} \tag{2.57}$$

Remark 2.5 An important feature to note about these Poisson brackets is the appearance of a non-ultralocal term in the third equation. In the known flat space integrable models, the presence of such a term is a good indicator for some breakdown of the conventional techniques at later stage (see e.g. [24] for exploring the fatal consequences of the non-ultralocal term in the PCM). However, in our model this term shows a surprisingly good behavior and in fact supports the entire further treatment.

Constraint algebra

We have already discussed that equations (2.23) do not descend from variation of the Lagrangian (2.45) but rather as constraints from its ancestor (2.7), i.e. before imposing conformal gauge (2.8). This structure is the same in the general class of coset space σ models introduced above.

Diffeomorphism invariance of (2.7) allows to bring the $2d$ metric $h_{\mu\nu}$ to conformal gauge (2.8). This gauge freedom is reflected in (2.7) by the fact that the components $T_{\pm\pm}$ of the $2d$ energy-momentum tensor arise as constraints with the unimodular parameters of $h_{\mu\nu}$ as Lagrange multipliers. In the language of canonical $2d$ gravity, these are the light-cone combinations of the Hamiltonian constraint (cf. (2.61) below) and the (one-dimensional)-diffeomorphism constraint; the associated Lagrange multipliers are the lapse and shift function of the two-dimensional (unimodular) metric [104]. In conformal gauge, these constraints read

$$T_{\pm\pm} \;\equiv\; 2\partial_\pm\rho\partial_\pm\hat{\sigma} - \rho\,\mathrm{tr}\,(P_\pm P_\pm) \;\approx\; 0\;. \tag{2.58}$$

After fixing the conformal gauge (2.8), the full model is thus given by the Lagrangian (2.45) and the conformal constraints $T_{\pm\pm}$. As first-class constraints the $T_{\pm\pm}$ generate the conformal transformations (2.10) of (2.45). With the canonical Poisson brackets (2.50), (2.57) we obtain:

$$\begin{aligned}
\{T_{\pm\pm}(x), \mathcal{V}(y)\} &= D_\pm\mathcal{V}\,\delta(x-y) \;=\; \mathcal{V}P_\pm\,\delta(x-y)\;, \qquad (2.59)\\
\{T_{\pm\pm}(x), P_\pm(y)\} &= \mp P_\pm(y)\,\delta'(x-y) + D_\pm P_\pm\,\delta(x-y)\;,\\
\{T_{\pm\pm}(x), P_\mp(y)\} &= D_\pm P_\pm\,\delta(x-y)\;,\\
\{T_{\pm\pm}(x), \rho(y)\} &= \partial_\pm\rho\,\delta(x-y)\;,\\
\{T_{\pm\pm}(x), \hat{\sigma}(y)\} &= \partial_\pm\hat{\sigma}\,\delta(x-y)\;,
\end{aligned}$$

where for the calculation of these equations one has to make use of the relations (2.40) as well as of the equations of motion (2.49). Thus, the transformations

$$\delta_{\xi^\pm}\varphi \;\equiv\; \int dx\,\xi^\pm(x)\,\{T_{\pm\pm}(x)\,,\,\varphi\} \;=\; -h^\pm_\varphi\,\partial_\pm\xi^\pm\,\varphi + \xi^\pm\,D_\pm\varphi\;, \tag{2.60}$$

reproduce (2.10) up to gauge transformations (2.41). The parameters $h^\pm_\varphi$ denote the conformal dimensions of the field φ. This formula illustrates the interplay between the canonical and the covariant framework. Canonically, the gauge parameter $\xi^\pm$ is defined as a function of and integrated over the spatial dimension x. Upon using the equations of motion for φ and restoring the time dependence of $\xi^\pm$ according to $\partial_\pm\xi^\mp = 0$, the r.h.s. of (2.60) takes a conformally covariant form. In particular, constant time translations are generated by integrating the Hamiltonian density

$$\mathcal{H} \;\equiv\; T_{++} + T_{--}\;, \tag{2.61}$$

over spatial x.

The conformal constraints $T_{\pm\pm}$ build two commuting copies of the classical Virasoro-Witt algebra

$$\begin{aligned} \{T_{\pm\pm}(x), T_{\pm\pm}(y)\} &= \mp\Big(T_{\pm\pm}(x) + T_{\pm\pm}(y)\Big)\,\delta'(x-y) \\ \{T_{\pm\pm}(x), T_{\mp\mp}(y)\} &= 0\;. \end{aligned} \tag{2.62}$$

In the course of applying the canonical formalism to (2.45), we have further encountered another set of constraints (2.55), having its origin in the **H**-gauge freedom (2.38). The Poisson algebra structure of the generators Φ_A is inherited from the algebra $\mathfrak{h}$:

$$\{\Phi_A(x), \Phi_B(y)\} \;=\; f_{AB}^{\;\;C}\,\Phi_C(x)\,\delta(x-y)\;. \tag{2.63}$$

In index-free notation (2.56) this reads

$$\Big\{\overset{1}{\Phi}(x)\;,\;\overset{2}{\Phi}(y)\Big\} \;=\; \Big[\Omega_{\mathfrak{h}}\;,\;\overset{2}{\Phi}(x)\Big]\;\delta(x-y)\;. \tag{2.64}$$

Under Φ the fields transform in an infinitesimal version of (2.38), (2.41):

$$\begin{aligned} \int dx\,\Big\{\mathrm{tr}\,(h(x)\Phi(x))\,,Q_1\Big\} &= \partial_1 h + [Q_1, h]\;, \\ \int dx\,\Big\{\mathrm{tr}\,(h(x)\Phi(x))\,,P_\pm\Big\} &= [P_\pm, h]\;. \end{aligned} \tag{2.65}$$

The conformal constraints $T_{\pm\pm}$ are invariant under Φ:

$$\{T_{\pm\pm}(x), \Phi(y)\} \;=\; 0\;. \tag{2.66}$$

In Dirac terminology [26] this means that all the constraints of the model are of the first class, thus compatible and responsible for gauge transformations. The full gauge algebra of constraints is given by (2.62), (2.63) and (2.66).

Remark 2.6 The action (2.65) of the constraints Φ does not describe the full gauge freedom observed in (2.38). According to the canonical formalism, h is just a function of the spatial coordinate x and thus carries only half of the gauge degrees of freedom of (2.38). Actually, the other half has been absorbed by the fact, that the field Q_0 from (2.39) has not shown up within the canonical framework. Hence, it appears decoupled from the rest of the theory and may be consistently put to zero.

Let us finally recall the possibility to fix the gauge algebra (2.62). As discussed in (2.20), the conformal transformations may be used to map the system (at least locally) to Weyl canonical coordinates, i.e. to identify the dilaton field ρ and its dual $\tilde{\rho}$ with the coordinates of the two-dimensional world-sheet. This is the precise analogue of adopting light-cone gauge in string theory [50]. Reference [3] gives an exhaustive discussion of this gauge fixing in the canonical treatment of models with cylindrical symmetry (2.21), handling all the physical boundary conditions with great care. In the following we will mainly – i.e. whenever necessary – stick to this particular choice of Weyl coordinates. Nonetheless, we will argue that the essential arising structures are to some extent generic.

3 Integrability

In this chapter, we exploit the integrability of the model (in technical terms: the existence of a linear system) to construct nonlocal integrals of motion from the associated transition matrices. We prove the stronger fact, that these conserved charges are invariant under the full gauge algebra (2.62), (2.63). In contrast to the nonlinear σ-model which allows a similar construction, there arise no ambiguities in the Poisson algebra of nonlocal charges here. Rather, as a central result we obtain the algebra (3.60), (3.61) which is closely related to the Yangian algebra known from various two-dimensional field theories [37, 10, 12]. This is analyzed in detail for the two particular choices of Weyl coordinates (2.20). The infinite dimensional symmetry group associated to these charges is revealed and their action on the physical fields is given. The Geroch algebra is recovered as the Lie-Poisson action of the algebra of $\mathfrak{g}$-valued functions on the complex plane. With some regularity assumptions on the fields the symmetry group acts transitively. Finally, we illustrate the results for the Abelian sector of the theory where due to linearization of the field equations the structures simplify essentially.

3.1 The linear system and the monodromy matrix

The model (2.45) is integrable in the sense that it possesses a linear system [7, 89]. I.e. the equations of motion (2.49) appear as integrability conditions of the following family of linear systems of differential equations, labeled by the spectral parameter γ:

$$\partial_\pm \widehat{\mathcal{V}}(x,t,\gamma) = \widehat{\mathcal{V}}(x,t,\gamma) L_\pm(x,t,\gamma) \ , \tag{3.1}$$

with

$$\widehat{\mathcal{V}}(x,t,\gamma) \in \mathbf{G} \ , \qquad L_\pm(x,t,\gamma) = Q_\pm + \frac{1 \mp \gamma}{1 \pm \gamma} P_\pm \in \mathfrak{g} \ .$$

In addition, the spectral parameter γ has to satisfy the differential equations

$$\gamma^{-1}\partial_\pm \gamma = \frac{1 \mp \gamma}{1 \pm \gamma} \rho^{-1} \partial_\pm \rho \ , \tag{3.2}$$

which due to (2.46) are compatible and have the general solution

$$\gamma(x,t,w) = \frac{1}{\rho}\left(w + \tilde{\rho} - \sqrt{(w+\tilde{\rho})^2 - \rho^2} \right) \ , \tag{3.3}$$

with a constant of integration w. This constant may be understood as the underlying constant spectral parameter of (3.1); in contrast we will refer to γ as the variable spectral parameter.

Remark 3.1 The original currents contained in $L_\pm$ (3.1) determine $\widehat{\mathcal{V}}$ only up to left multiplication with a matrix depending on the constant spectral parameter w:

$$\widehat{\mathcal{V}}(x,t,\gamma) \mapsto S(w)\widehat{\mathcal{V}}(x,t,\gamma) \ , \quad \text{with } \ S(w) \in \mathbf{G} \ . \tag{3.4}$$

Later on we will encounter different possibilities how to eliminate this freedom.

Remark 3.2 The nonlinear σ-model admits a similar linear system with constant spectral parameter γ [107, 121]. The coordinate dependence of γ in (3.1) turns out to be essential for the entire following treatment, here. Its origin lies in the explicit appearance of the dilaton field ρ in (2.49).

The spectral parameters

Here, we collect some useful formulas illustrating the interplay between the variable and the constant spectral parameters γ and w.

The parameter γ lives on the Riemann surface defined by $\sqrt{(w+\tilde{\rho}+\rho)(w+\tilde{\rho}-\rho)}$, which is a twofold covering of the complex w-plane with x^μ-dependent branch-cut. Transition between the two sheets is performed by to $\gamma \mapsto \frac{1}{\gamma}$. The branch-cut connects the points $w = -\tilde{\rho} \pm \rho$ on the real w-axis, which correspond to $\gamma(w = -\tilde{\rho} \pm \rho) = \pm 1$. The real w with $|w + \tilde{\rho}| < |\rho|$ are mapped onto the unit circle $|\gamma| = 1$. Real w with $|w + \tilde{\rho}| > |\rho|$ are mapped onto the real γ-axis. The image of the axis $\Re(w) = -\tilde{\rho}$ is the imaginary axis in the γ-plane.

Dividing the w-plane into two regions $H_\pm$ and the γ-plane into four regions $D_\pm, \tilde{D}_\pm$ according to Fig. 1, $D_\pm$ and $\tilde{D}_\pm$ lie over $H_\pm$, respectively.

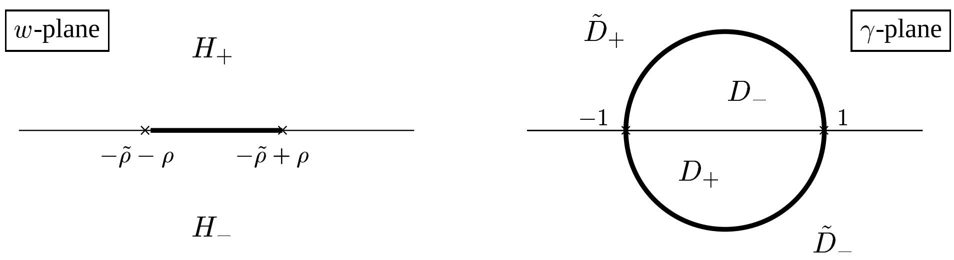


Figure 1: The spectral parameter planes

Remark 3.3 It is important that for fixed $w \notin \mathbb{R}$ and continously varying ρ and $\tilde{\rho}$, the parameter γ does not cross the boundaries which separate these regions. The limits of its trajectories are given by

$$\gamma(\rho \to 0) \ \to \ \begin{cases} 0 \\ \infty \end{cases} , \quad \gamma(\rho \to \infty) \ \to \ \begin{cases} \mathrm{i} \\ -\mathrm{i} \end{cases} , \quad \gamma(\tilde{\rho} \to \pm\infty) \ \to \ \begin{cases} 0 \\ \infty \end{cases} , \tag{3.5}$$

where the two values correspond to the two sheets of γ.

Another useful formula is the inverse expression $w(\gamma) = \frac{1}{2}\rho\,(\gamma + \frac{1}{\gamma}) - \tilde{\rho}$, which e.g. implies

$$\gamma^{-1}\partial_w\gamma \;=\; -\frac{2\gamma}{\rho(1-\gamma^2)}\;. \tag{3.6}$$

Two spectral parameters $\gamma(x,t,v)$ and $\gamma(x,t,w)$ at coinciding coordinates x,t are related by:

$$v - w \;=\; \frac{\rho}{2}\,\frac{(\gamma(v)-\gamma(w))\,(\gamma(v)\gamma(w)-1)}{\gamma(v)\gamma(w)}\;. \tag{3.7}$$

Monodromy matrix

The involution τ which according to (2.37) defines the symmetric space $\mathbf{G}/\mathbf{H}$ can be extended to an involution τ^∞ which acts on $\mathbf{G}$-valued functions of the spectral parameter γ by combining the action on $\mathbf{G}$ with a transition between the two sheets of γ [61]:

$$\tau^\infty\Big(\widehat{\mathcal{V}}(\gamma)\Big) \equiv \tau\left(\widehat{\mathcal{V}}(\tfrac{1}{\gamma})\right)\;. \tag{3.8}$$

This generalized involution leaves the connection $L_\pm(\gamma)$ of the linear system (3.1) invariant. Thus, it motivates the following definition [13]:

$$\mathcal{M}(\gamma) \;\equiv\; \widehat{\mathcal{V}}(\gamma)\;\tau^\infty\Big(\widehat{\mathcal{V}}^{-1}(\gamma)\Big) \;=\; \widehat{\mathcal{V}}(\gamma)\,\tau\Big(\widehat{\mathcal{V}}^{-1}(\tfrac{1}{\gamma})\Big)\;. \tag{3.9}$$

The matrix $\mathcal{M}$ is called the monodromy matrix associated with $\widehat{\mathcal{V}}(\gamma)$. Due to the invariance of $L_\pm(\gamma)$ under τ^∞, the linear system (3.1) implies

$$\partial_\pm\mathcal{M} = 0 \quad\Rightarrow\quad \mathcal{M} = \mathcal{M}(w)\;, \tag{3.10}$$

thus $\mathcal{M}$ depends on the constant spectral parameter w only. Its independence of the coordinates in particular implies, that the monodromy matrix does not feel the $x^\pm$-dependent branch-cut of Figure 1.

According to Remark 3.1, the monodromy $\mathcal{M}$ is defined only up to the conjugation

$$\mathcal{M}(w) \;\mapsto\; S(w)\mathcal{M}(w)\;\tau\big(S^{-1}(w)\big)\;,$$

with some $S(w) \in \mathbf{G}$. A preferred choice of eliminating this freedom has been introduced by Breitenlohner and Maison [13] by demanding holomorphy of $\widehat{\mathcal{V}}(\gamma)$ inside a domain in the γ-plane containing the unit disc $D_+ \cup D_-$. [4] This uniquely fixes $\widehat{\mathcal{V}}$ up to a constant matrix. Whenever necessary, we will denote the corresponding solution of (3.1) by $\widehat{\mathcal{V}}_{\text{BM}}$. The absence of singularities in the disc in particular allows to recover the original field $\mathcal{V}$ via

$$\mathcal{V}(x) = \widehat{\mathcal{V}}_{\text{BM}}(x,\gamma)|_{\gamma=0}\;. \tag{3.11}$$

The corresponding monodromy matrix

$$\mathcal{M}_{\text{BM}}(w(\gamma)) \;=\; \widehat{\mathcal{V}}_{\text{BM}}(\gamma)\;\tau^\infty\Big(\widehat{\mathcal{V}}_{\text{BM}}^{-1}(\gamma)\Big) \;=\; \widehat{\mathcal{V}}_{\text{BM}}(\gamma)\;\tau\Big(\widehat{\mathcal{V}}_{\text{BM}}^{-1}(\tfrac{1}{\gamma})\Big) \tag{3.12}$$

[4] Roughly speaking, the invariance $w(\gamma) = w(\gamma^{-1})$ allows to reflect all singularities at the unit circle by multiplying $\widehat{\mathcal{V}}$ with a suitable $S(w)$.

is non-singular as a function of γ in an annular region containing the unit circle $|\gamma| = 1$. The matrix $\widehat{\mathcal{V}}_{\text{BM}}(\gamma)$ may then be recovered from $\mathcal{M}_{\text{BM}}$ by solving (3.12) as a (generalized) Riemann-Hilbert factorization problem on this annulus. Thus, $\mathcal{M}_{\text{BM}}$ contains the complete information about $\widehat{\mathcal{V}}$. Since it obeys

$$\mathcal{M}_{\text{BM}}(w) = \tau^{\infty}\big(\mathcal{M}_{\text{BM}}^{-1}(w)\big) = \tau\left(\mathcal{M}_{\text{BM}}^{-1}(w)\right) \ , \tag{3.13}$$

it can be represented as

$$\mathcal{M}_{\text{BM}}(w) = S_{\text{BM}}(w)\ \tau\big(S_{\text{BM}}^{-1}(w)\big) \ . \tag{3.14}$$

This implies that $\widehat{\mathcal{V}}_{\text{BM}}$ factorizes into

$$\widehat{\mathcal{V}}_{\text{BM}}(\gamma(w)) = S_{\text{BM}}(w)\widehat{\mathcal{V}}_{\text{BZ}}(\gamma(w)) \ , \tag{3.15}$$

with a matrix $\widehat{\mathcal{V}}_{\text{BZ}}(\gamma(w))$ which also solves the linear system (3.1). Its associated monodromy (3.9) vanishes, i.e.

$$\widehat{\mathcal{V}}_{\text{BZ}}(\gamma) = \tau^{\infty}\Big(\widehat{\mathcal{V}}_{\text{BZ}}(\gamma)\Big) \ . \tag{3.16}$$

This solution of (3.1) has been used in the approach of Belinskii and Zakharov [7]. It is defined up to left multiplication with $\mathbf{H}$-valued matrices $S(w)$ (for which $\tau(S)=S$).

3.2 Transition matrices and their Poisson algebra

The monodromy matrix $\mathcal{M}_{\text{BM}}$, introduced in the previous paragraph, apparently is a good candidate for generating nontrivial integrals of motion. At least in principle, it carries the entire information about the original fields $\mathcal{V}$. However, so far its usage as a canonical object suffers from the fact that its definition is a rather implicit one, involving the holomorphy of $\widehat{\mathcal{V}}_{\text{BM}}$ in the unit γ-disc. A priori, it is not clear how to explicitly construct this object from given fields $\mathcal{V}$, thus we miss the information about the symplectic structure of the encoded integrals of motion. However, in the next section we will be able to identify $\mathcal{M}_{\text{BM}}$ in the canonical framework (cf. (3.40), (3.49), below). In this section, we introduce the transition matrices of (3.1) as canonical objects. We extract the encoded integrals of motion and derive their Poisson algebra.

The transition matrices associated to the linear system (3.1) are defined by

$$\begin{aligned} U(x,y,t,w) \ &\equiv\ \widehat{\mathcal{V}}^{-1}(x,t,\gamma(x,t,w))\ \widehat{\mathcal{V}}(y,t,\gamma(y,t,w)) \\ &=\ \mathcal{P}\exp\int_x^y dz\ L_1(z,t,\gamma(z,t,w)) \ , \end{aligned} \tag{3.17}$$

which are unique functionals of the connection $L_{\pm} = \frac{1}{2}\left(L_0 \pm L_1\right)$. The integrand in (3.17) lives on the twofold covering of the complex w-plane with a branch cut which according to Figure 1 varies on the real w-axis while z runs from x to y. Having in mind Remark 3.3, the transition matrix $U(x,y,t,w)$ is well defined for $w \notin \mathbb{R}$. It also lives on the twofold covering of the w-plane and like $L_{\pm}$ it is invariant under the generalized involution τ^{∞} introduced in (3.8). In other words, $U(x,y,t,w)$ is completely determined by its values on one of the sheets; its values on the other sheet are given by $\tau(U(x,y,t,w))$. Until explicitly stated, we shall in the following always consider the sheet with $\gamma \in D_+ \cup D_-$ inside the unit disc.

The values of $U(x,y,t,w)$ on the real w-axis can be obtained from evaluating the limit

$$\lim_{\epsilon\to 0} U(x,y,t,w\pm \mathrm{i}\epsilon) \qquad \text{with} \;\; \epsilon \in \mathbb{R}_{>0} \;, \tag{3.18}$$

which may however give two different results for $+$ and $-$.

Integrals of motion

Inspecting the time dependence of the transition matrices we can conclude how to extract integrals of motion. Namely, the modified transition matrices

$$\widetilde{U}(x,y,t,w) \equiv \mathcal{V}(x) U(x,y,t,w) \mathcal{V}^{-1}(y) \;, \tag{3.19}$$

satisfy

$$\partial_t \widetilde{U}(x,y,t,w) \;=\; -\widetilde{L}_0(x,t,\gamma(x,t,w))\, \widetilde{U} + \widetilde{U}\, \widetilde{L}_0(y,t,\gamma(y,t,w)) \;, \tag{3.20}$$

with

$$\widetilde{L}_0 \;=\; \mathcal{V} L_0 \mathcal{V}^{-1} - \partial_0 \mathcal{V} \mathcal{V}^{-1} \;=\; \frac{2\gamma^2}{1-\gamma^2}\, \mathcal{V} P_0 \mathcal{V}^{-1} - \frac{2\gamma}{1-\gamma^2}\, \mathcal{V} P_1 \mathcal{V}^{-1} \;.$$

There are now several possibilities to construct integrals of motion:

- Assuming periodic boundary conditions for P_0 and P_1 on an interval $[-\frac{L}{2}, \frac{L}{2}]$, (3.20) shows that the eigenvalues of $\widetilde{U}(-\frac{L}{2}, \frac{L}{2}, t, w)$ are time-independent if also ρ and $\tilde{\rho}$ are periodic functions in x. Charges of this type have been studied in [91]. In general however, assuming periodic boundary conditions on the physical fields P_0, P_1 and ρ does not guarantee periodicity of the dual field $\tilde{\rho}$ defined by (2.18). The variable spectral parameter γ then is not periodic in x, and it remains an open problem how to extract proper integrals of motion from $\widetilde{U}$. This is an essential difference to the normal integrable systems with constant spectral parameter.

- The transition matrix $\widetilde{U}(x_0, y_0, t, w)$ itself becomes an integral of motion if

  $$L_0(x_0,t,\gamma(x_0,t,w)) = L_0(y_0,t,\gamma(y_0,t,w)) = 0 \;. \tag{3.21}$$

  According to the form of L_0 this happens in two cases:

  – $$P_0(x_0) = P_1(x_0) = 0 \quad \text{and} \quad \gamma(x_0) \neq \pm 1 \;, \tag{3.22}$$

  – $$\gamma(x_0) = 0 \quad \text{and} \quad |P_0(x_0)| < \infty \;,\; |P_1(x_0)| < \infty \;, \tag{3.23}$$

  and accordingly for y_0. The first case (3.22) e.g. occurs for asymptotically vanishing currents with $|x_0| \to \infty$. This may describe asymptotically Minkowskian spacetimes (cf. (2.28)).

  The second case (3.23) is even more interesting since it makes use of the field dependence of the variable spectral parameter. According to (3.5) the crucial limits at which γ tends to zero are $\rho \to 0$ and $\tilde{\rho} \to \pm\infty$. The interpolating transition matrices thus provide integrals of motion.

- If there is at least one point x_0 in spacetime, where according to (3.22) or (3.23) $L_0(x_0, t, \gamma(x_0, t, w))$ vanishes, the transition matrix

$$\widehat{\mathcal{V}}_{x_0}(x, t, \gamma(x, t, w)) \equiv \mathcal{V}(x_0, t) U(x_0, x, t, w) \tag{3.24}$$

  forms a solution of the linear system (3.1). We can then further extract its monodromy matrix (3.9) as a canonical object, which itself is an integral of motion.

What is still missing of course is the degree of nontriviality of all these integrals of motion. Assume e.g. that we had identified a solution $\widehat{\mathcal{V}}_{\text{BZ}}$ in (3.24) then according to (3.16) its monodromy matrix would carry no information at all. The content of the integrals of motion will thus have to be checked separately whenever in the following we will construct integrals of motion according to the procedure described above.

Conformal invariance

So far we have just shown, that certain transition matrices constructed from (3.19) are integrals of motion, i.e. conserved in time. Constant time translation is generated by the integral over the Hamiltonian density (2.61) (in the language of canonical gravity: by the Hamiltonian constraint integrated with a constant lapse function). In fact, meaningful observables in the sense of Dirac should satisfy much more, namely be invariant under the full gauge algebra (2.62), (2.63). In this paragraph we show that this is indeed the case for the integrals of motion obtained above.

First, we check the transformation behavior of the modified transition matrices $\widetilde{U}$ under the $\mathbf{H}$-gauge transformations (2.65). It is

$$\left\{ \Phi(z) \, , \widetilde{U}(x, y, w) \right\} \;=\; 0 \;, \tag{3.25}$$

i.e. the modified transition matrices are $\mathbf{H}$-singlets for arbitrary endpoints x and y. This mainly distinguishes them from the normal transition matrices (3.17), which transform by conjugation. The transformation behavior under the conformal constraints $T_{\pm\pm}$ may be obtained from the general formula (3.29) below and yields

$$\begin{aligned} \left\{ T_{\pm\pm}(z) \, , \widetilde{U}(x_0, y_0, w) \right\} \;&=\; -\widetilde{L}_{\pm}(x_0) \widetilde{U}(x_0, y_0, w)\, \delta(z - x_0) \\ &\qquad + \widetilde{U}(x_0, y_0, w) \widetilde{L}_{\pm}(y_0)\, \delta(z - y_0) \;, \end{aligned} \tag{3.26}$$

with

$$\widetilde{L}_{\pm} \;\equiv\; \mathcal{V} L_{\pm} \mathcal{V}^{-1} - \partial_{\pm} \mathcal{V} \mathcal{V}^{-1} \;=\; \mp \frac{2\gamma}{1 \pm \gamma}\, \mathcal{V} P_{\pm} \mathcal{V}^{-1} \;.$$

This is the direct generalization of (3.20). The r.h.s. of (3.26) vanishes under the very same conditions on x_0, y_0 that were discussed for (3.20). I.e. all the integrals of motion obtained in the previous section are indeed invariant under arbitrary conformal transformations, generated bt the $T_{\pm\pm}$.

Let us finally compute the Poisson bracket between the integrals of motion and the conformal factor σ. An arbitrary transition matrix (3.19) satisfies

$$\left\{\widetilde{U}(x,y,v)\,,\,\partial_1\sigma(z)\right\} \;=\; -\widetilde{U}(x,z,v)\,\partial_v\widetilde{L}_1(x,\gamma(v))\,\widetilde{U}(z,y,v)\;, \tag{3.27}$$

which in turn follows from (2.50), (3.29) and the fact that $\partial_v L_1 = \partial_{\tilde{\rho}} L_1$. By integration we obtain

$$\left\{\widetilde{U}(x_0,y_0,v)\,,\,(\sigma(y_0)-\sigma(x_0))\right\} \;=\; -\partial_v\widetilde{U}(x_0,y_0,v)\;, \tag{3.28}$$

using that the connection $\widetilde{L}_1$ vanishes at the critical points $x_0,\ y_0$. Thus we see, how the conformal factor σ at the spatial boundaries provides a derivation operator of the integrals of motion.

Poisson algebra of transition matrices

This paragraph is devoted to the (rather technical) calculation of the Poisson brackets between two transition matrices with pairwise distinct endpoints. A similar calculation has been done for the PCM [24]. The results however differ in two essential points. First, the underlying coset structure here implies the appearance of a twist in the resulting Poisson algebra (3.46), (3.47). Second, the calculation for the PCM is obstructed by certain ambiguities which arise due to the non-ultralocal contributions of the original Poisson brackets (2.57). They prevent a well-defined answer for the Poisson brackets between transition matrices with coinciding endpoints. In particular this spoils the Poisson algebra of transition matrices relating the spatial boundaries. In our model on the other hand, the coordinate dependence of the spectral parameter – caused by the coupling of the dilaton field ρ in (2.49) – yields an intrinsic regularization of these ambiguities at the spatial boundaries [77] provided that we assume the proper asymptotic behavior of the fields ρ and $\tilde{\rho}$. We shall describe this in detail.

Let $U(x,y,v)$ and $U(x',y',w)$ be the transition matrices with spectral parameters v and w, respectively, and pairwise distinct endpoints x,y and x',y'.[5] The definition (3.17) implies the relations [39]

$$\{\,U(x,y,v)\,,\,X\,\} \;=\; \int_x^y dz\; U(x,z,v)\,\{\,L_1(z,\gamma_1)\,,\,X\,\}\,U(z,y,v)\;, \tag{3.29}$$

for an arbitrary function X.and

$$\begin{aligned}\left\{\overset{1}{U}(x,y,v)\,,\,\overset{2}{U}(x',y',w)\right\} \;=\;& \int_x^y dz\int_{x'}^{y'} dz'\;\left(\overset{1}{U}(x,z,v)\,\overset{2}{U}(x',z',w)\right)\times \\ & \left\{\overset{1}{L}_1(z,\gamma_1)\,,\overset{2}{L}_1(z',\gamma_2)\right\}\left(\overset{1}{U}(z,y,v)\,\overset{2}{U}(z',y',w)\right)\,,\end{aligned} \tag{3.30}$$

with $\gamma_1\equiv\gamma(z,v)$, $\gamma_2\equiv\gamma(z',w)$.

[5] For clarity, we drop the coinciding argument t throughout this calculation. Nonetheless, so far all the arising objects are time-dependent.

Due to the coset structure of the model, it is a priori not obvious, that the Poisson algebra of the connection L_1 of the linear system (3.1) is of a closed form. However, this turns out to be true on the constraint surface (2.55):

$$\left\{ \overset{1}{L}_1(z,\gamma_1) \ , \ \overset{2}{L}_1(z',\gamma_2) \right\} = \tag{3.31}$$
$$\begin{aligned}
&-\frac{2\gamma_1\gamma_2}{\rho(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)} \left[\Omega_{\mathfrak{h}} \ , \ \overset{1}{L}_1(\gamma_1)+\overset{2}{L}_1(\gamma_2) \right] \delta(z-z') \\
&-\frac{2\gamma_2^2(1-\gamma_1^2)}{\rho(1-\gamma_2^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)} \left[\Omega_{\mathfrak{k}} \ , \ \overset{1}{L}_1(\gamma_1) \right] \delta(z-z') \\
&-\frac{2\gamma_1^2(1-\gamma_2^2)}{\rho(1-\gamma_1^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)} \left[\Omega_{\mathfrak{k}} \ , \ \overset{2}{L}_1(\gamma_2) \right] \delta(z-z') \\
&-\frac{2\,\Omega_{\mathfrak{k}}}{(1-\gamma_1^2)(1-\gamma_2^2)} \left(\frac{\gamma_1(1+\gamma_2^2)}{\rho(z)} + \frac{\gamma_2(1+\gamma_1^2)}{\rho(z')} \right) \partial_z \delta(z-z') \ .
\end{aligned}$$

Inserting this into (3.30) and using (3.7) and definition (3.17) leads to

$$\left\{ \overset{1}{U}(x,y,v) \ , \ \overset{2}{U}(x',y',w) \right\} = \tag{3.32}$$
$$\begin{aligned}
&-\int_x^y dz \int_{x'}^{y'} dz' \ \frac{1}{v-w} \ \delta(z-z') \ (\partial_z + \partial_{z'}) \, \Xi_{\mathfrak{h}} \\
&-\int_x^y dz \int_{x'}^{y'} dz' \ \frac{2\gamma_2^2(1-\gamma_1^2)}{\rho(1-\gamma_2^2)(\gamma_2-\gamma_1)(1-\gamma_1\gamma_2)} \ \delta(z-z') \ \partial_z \Xi_{\mathfrak{k}} \\
&+\int_x^y dz \int_{x'}^{y'} dz' \ \frac{2\gamma_1^2(1-\gamma_2^2)}{\rho(1-\gamma_1^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)} \ \delta(z-z') \ \partial_{z'} \Xi_{\mathfrak{k}} \\
&-\int_x^y dz \int_{x'}^{y'} dz' \ \frac{2\left(\rho^{-1}(z)\gamma_1(1+\gamma_2^2) + \rho^{-1}(z')\gamma_2(1+\gamma_1^2)\right)}{(1-\gamma_1^2)(1-\gamma_2^2)} \ \Xi_{\mathfrak{k}} \ \partial_z \delta(z-z') \ ,
\end{aligned}$$

with

$$\begin{aligned}
\Xi_{\mathfrak{h}} &\equiv \overset{1}{U}(x,z,v) \overset{2}{U}(x',z',w) \ \Omega_{\mathfrak{h}} \ \overset{1}{U}(z,y,v) \overset{2}{U}(z',y',w) \ , \\
\Xi_{\mathfrak{k}} &\equiv \overset{1}{U}(x,z,v) \overset{2}{U}(x',z',w) \ \Omega_{\mathfrak{k}} \ \overset{1}{U}(z,y,v) \overset{2}{U}(z',y',w) \ .
\end{aligned}$$

Partial integration of the first three terms reduces the expression to boundary terms. There arise additional terms from derivatives of the spectral parameter (cf. (3.2)). E.g. the second term in (3.32) gives a contribution of

$$\begin{aligned}
&-\frac{8\gamma_1^2\gamma_2^2\left((\gamma_1-\gamma_2)^2 + (1-\gamma_1\gamma_2)^2\right)}{(1-\gamma_1^2)(1-\gamma_2^2)(\gamma_1-\gamma_2)^2(1-\gamma_1\gamma_2)^2} \ \rho^{-2}\partial_0\rho \\
&+\frac{\gamma_1(1+\gamma_1^2)\gamma_2^2\left(4\gamma_1(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2) + 2(1-\gamma_1^2)(1-2\gamma_1\gamma_2+\gamma_2^2)\right)}{(\gamma_1-\gamma_2)^2(1-\gamma_1\gamma_2)^2(1-\gamma_1^2)(1-\gamma_2^2)} \ \rho^{-2}\partial_1\rho \ ;
\end{aligned}$$

the third term yields the same with opposite sign and γ_1 and γ_2 interchanged. This combines into a term proportional to $\rho^{-2}\partial_1\rho$ which is precisely cancelled by the contribution from the last term in (3.32) (note the different arguments of the dilaton ρ). Altogether, there remain the following boundary terms

$$\left\{\overset{1}{U}(x,y,v)\,,\,\overset{2}{U}(x',y',w)\right\} = \tag{3.33}$$

$$\begin{aligned}
\frac{1}{v-w} \times \Big\{ & \theta(x,x',y)\left(\overset{1}{U}(x,x',v)\;\Omega_{\mathfrak{h}}\;\overset{1}{U}(x',y,v)\overset{2}{U}(x',y',w)\right)\\
&+\theta(x',x,y')\left(\overset{2}{U}(x',x,w)\;\Omega_{\mathfrak{h}}\;\overset{1}{U}(x,y,v)\overset{2}{U}(x,y',w)\right)\\
&-\theta(x,y',y)\left(\overset{1}{U}(x,y',v)\overset{2}{U}(x',y',w)\;\Omega_{\mathfrak{h}}\;\overset{1}{U}(y',y,v)\right)\\
&-\theta(x',y,y')\left(\overset{1}{U}(x,y,v)\overset{2}{U}(x',y,w)\;\Omega_{\mathfrak{h}}\;\overset{2}{U}(y,y',w)\right)\Big\}\\
+\frac{\theta(x,x',y)}{v-w}&\left(\overset{1}{U}(x,x',v)\;\Omega_{\mathfrak{k}}\;\overset{1}{U}(x',y,v)\overset{2}{U}(x',y',w)\right)\frac{\gamma(x',v)(1-\gamma^2(x',w))}{\gamma(x',w)(1-\gamma^2(x',v))}\\
+\frac{\theta(x',x,y')}{v-w}&\left(\overset{2}{U}(x',x,w)\;\Omega_{\mathfrak{k}}\;\overset{1}{U}(x,y,v)\overset{2}{U}(x,y',w)\right)\frac{\gamma(x,w)(1-\gamma^2(x,v))}{\gamma(x,v)(1-\gamma^2(x,w))}\\
-\frac{\theta(x,y',y)}{v-w}&\left(\overset{1}{U}(x,y',v)\overset{2}{U}(x',y',w)\;\Omega_{\mathfrak{k}}\;\overset{1}{U}(y',y,v)\right)\frac{\gamma(y',v)(1-\gamma^2(y',w))}{\gamma(y',w)(1-\gamma^2(y',v))}\\
-\frac{\theta(x',y,y')}{v-w}&\left(\overset{1}{U}(x,y,v)\overset{2}{U}(x',y,w)\;\Omega_{\mathfrak{k}}\;\overset{2}{U}(y,y',w)\right)\frac{\gamma(y,w)(1-\gamma^2(y,v))}{\gamma(y,v)(1-\gamma^2(y,w))}\;,
\end{aligned}$$

where we have made use of the abbreviation:

$$\theta(x,y,z) = \begin{cases} 1 & \text{for } x<y<z \\ 0 & \text{else } \;(x\neq y\neq z) \end{cases} .$$

We are mainly interested in the modified transition matrices from (3.19). Their Poisson brackets acquire additional contributions from

$$\begin{aligned}
\left\{\overset{1}{U}(x,y,v)\,,\,\overset{2}{\mathcal{V}}(x')\right\} &= \frac{2\,\theta(x,x',y)\,\gamma(x',v)}{\rho(x')(1-\gamma^2(x',v))}\;\overset{1}{U}(x,x',v)\,\overset{2}{\mathcal{V}}(x')\,\Omega_{\mathfrak{k}}\,\overset{1}{U}(x',y,v)\;,\\
\left\{\overset{1}{\mathcal{V}}(x)\,,\,\overset{2}{U}(x',y',w)\right\} &= -\frac{2\,\theta(x',x,y')\,\gamma(x,w)}{\rho(x)(1-\gamma^2(x,w))}\;\overset{2}{U}(x',x,w)\,\overset{1}{\mathcal{V}}(x)\,\Omega_{\mathfrak{k}}\,\overset{2}{U}(x,y',w)\;,
\end{aligned}$$

etc.

The final result then is

$$\overset{1}{\mathcal{V}}{}^{-1}(x)\,\overset{2}{\mathcal{V}}{}^{-1}(x')\left\{\overset{1}{\widetilde{U}}(x,y,v)\;,\;\overset{2}{\widetilde{U}}(x',y',w)\right\}\overset{1}{\mathcal{V}}(y)\,\overset{2}{\mathcal{V}}(y') \tag{3.34}$$

$$\begin{aligned}
= \quad \frac{1}{v-w}\;\times\;\Big\{\; & \theta(x,x',y)\,\Big(\overset{1}{U}(x,x',v)\,\Omega_{\mathfrak{h}}\;\overset{1}{U}(x',y,v)\,\overset{2}{U}(x',y',w)\Big) \\
& + \theta(x',x,y')\,\Big(\overset{2}{U}(x',x,w)\;\;\Omega_{\mathfrak{h}}\;\;\overset{1}{U}(x,y,v)\,\overset{2}{U}(x,y',w)\Big) \\
& - \theta(x,y',y)\,\Big(\overset{1}{U}(x,y',v)\;\overset{2}{U}(x',y',w)\;\;\Omega_{\mathfrak{h}}\;\;\overset{1}{U}(y',y,v)\Big) \\
& - \theta(x',y,y')\,\Big(\overset{1}{U}(x,y,v)\;\overset{2}{U}(x',y,w)\;\;\Omega_{\mathfrak{h}}\;\;\overset{2}{U}(y,y',w)\Big)\;\Big\} \\
& + \frac{\theta(x,x',y)}{v-w}\,f(x',w,v)\,\Big(\overset{1}{U}(x,x',v)\,\Omega_{\mathfrak{k}}\,\overset{1}{U}(x',y,v)\,\overset{2}{U}(x',y',w)\Big) \\
& + \frac{\theta(x',x,y')}{v-w}\,f(x,v,w)\,\Big(\overset{2}{U}(x',x,w)\,\Omega_{\mathfrak{k}}\,\overset{1}{U}(x,y,v)\,\overset{2}{U}(x,y',w)\Big) \\
& - \frac{\theta(x,y',y)}{v-w}\,f(y',w,v)\,\Big(\overset{1}{U}(x,y',v)\,\overset{2}{U}(x',y',w)\,\Omega_{\mathfrak{k}}\,\overset{1}{U}(y',y,v)\Big) \\
& - \frac{\theta(x',y,y')}{v-w}\,f(y,v,w)\,\Big(\overset{1}{U}(x,y,v)\,\overset{2}{U}(x',y,w)\,\Omega_{\mathfrak{k}}\,\overset{2}{U}(y,y',w)\Big)\;,
\end{aligned}$$

with θ from above and

$$f(x,v,w)\equiv\frac{1-2\gamma(x,w)\gamma(x,v)+\gamma^2(x,w)}{1-\gamma^2(x,w)}\;.$$

This result superficially resembles the corresponding bracket arising in the PCM [24]. In fact, neglecting the coset structure (i.e. formally putting $\Omega_{\mathfrak{h}}=\Omega_{\mathfrak{k}}=\Omega_{\mathfrak{g}}$) and dropping the coordinate dependence of the spectral parameters λ, equation (3.34) explicitly reduces to the brackets appearing in the PCM.

At first sight, we thus face the same fatal problem: With distinct endpoints x,x',y,y' the algebra (3.34) is uniquely and well defined, satisfying in particular antisymmetry and Jacobi identities. The limit to coinciding endpoints on the other hand is obviously ambiguous. E.g. it is easy to check that

$$\lim_{\substack{x\to x'\\ x>x'}}\left\{\overset{1}{\widetilde{U}}(x,y,v)\;,\;\overset{2}{\widetilde{U}}(x',y',w)\right\}\;\neq\;\lim_{\substack{x\to x'\\ x<x'}}\left\{\overset{1}{\widetilde{U}}(x,y,v)\;,\;\overset{2}{\widetilde{U}}(x',y',w)\right\}\;,$$

since

$$f(x,v,w)\neq f(x,w,v)\;. \tag{3.35}$$

In the PCM this ambiguity survives in the limit $x,x'\to-\infty,\;y,y'\to\infty$ with no possibility to cure this in accordance with antisymmetry and validity of the Jacobi identities [86, 24]. The corresponding transition matrices relating the spatial boundaries however are the main objects of interest, since they encode the integrals of motion. Several procedures have been suggested to nevertheless make sense out of the classical Poisson algebra of the PCM [38, 31, 87].

In our model on the other hand, the coordinate dependence of the spectral parameter γ changes the situation drastically. Namely, since the function $f(x, v, w)$ inherits this dependence, the ambiguity (3.35) may "fade out" in a certain limit. This happens if at the endpoints x, x' and y, y' the variable spectral parameter γ becomes independent of the constant one, such that (3.35) becomes an equality. These possible fixpoints of the spectral parameter are 0, ∞ and $\pm\mathrm{i}$ (cf. (3.5), (3.6)). In this case equation (3.34) shrinks to an algebra related to Drinfeld's Yangian [28]. We shall demonstrate this for the two choices of Weyl coordinates (2.20) in the next section.

3.3 Nonlocal charges and their Poisson algebra

In this section, we analyze the integrals of motion obtained above for the two particular cases of Weyl coordinates (2.20) assuming the vector field $\partial_\mu\rho$ to be globally space- and timelike, respectively. Evaluating the general result (3.34), we obtain the relevant Poisson algebra of nonlocal charges. The same fundamental structures arise from somewhat different sides.

Nonlocal charges for a spacelike (radial) dilaton

For this paragraph, let us assume that the vector field $\partial_\mu\rho$ is globally spacelike. We can then identify ρ with a radial coordinate $x = r \in [0, \infty[$. This is a common coordinate system for the description of cylindrically symmetric gravitational waves [69, 79, 3]; the symmetry axis is given by $x = 0$. For pure Einstein gravity, we have already introduced these coordinates in (2.21). The dual field $\tilde{\rho}$ is identified with the time:

$$\rho \equiv x \in [0, \infty[\ , \qquad \tilde{\rho} \equiv t\ . \tag{3.36}$$

Let the physical currents P_0, P_1 fall off sufficiently fast at spatial infinity $x \to \infty$ with $\mathcal{V} \to I$ and behave regularly on the axis $x = 0$. According to (2.28), in four dimensions we can demand this for the currents which are either related to the original matrix $\mathcal{V}$ from (2.25) or to the matrix $\mathcal{V}_D$ carrying some of the the dual potentials. A physically interesting class of gravitational waves is e.g. described by restricting to regular $\mathcal{V}_D$ on the symmetry axis [17].

In the sense of (3.22), (3.23) there are thus two interesting points : $x = \infty$ satisfying (3.22) and $x = 0$ with (3.23). According to (3.24) they give rise to the following two solutions of the linear system:

$$\begin{aligned} \widehat{\mathcal{V}}_0(x, \gamma(x, w)) &\equiv \mathcal{V}(0)\, U(0, x, w) \;=\; \widehat{\mathcal{V}}_{\mathrm{BM}}(x, \gamma(x, w))\ , \\ \widehat{\mathcal{V}}_\infty(x, \gamma(x, w)) &\equiv \mathcal{V}(\infty)\, U(\infty, x, w)\ . \end{aligned} \tag{3.37}$$

The second equality in (3.37) follows from the behavior of the moving branch cut (cf. Figure 1) in $\widehat{\mathcal{V}}_0(\gamma)$. The matrix $\widehat{\mathcal{V}}_0$ is the (unique) solution of (3.1) which as a function of γ is holomorphic in the unit disc $D_+ \cup D_-$ and thus coincides with $\widehat{\mathcal{V}}_{\mathrm{BM}}$ from (3.12). The solution $\widehat{\mathcal{V}}_\infty(\gamma)$ on the other hand is the unique one which is holomorphic in the lower half plane $D_+ \cup \tilde{D}_-$ or the upper half plane $D_- \cup \tilde{D}_+$, respectively, depending on the sign of $\Im w$. In particular, $\widehat{\mathcal{V}}_\infty(\gamma(w))$ as a function of w is discontinuous along the real w-axis since for $x \to \infty$ the branch cut blows up and cuts the w plane into two halves (cf. (3.18)).

From (3.37) we extract the integrals of motion

$$U_{\pm}(w) \equiv \widehat{\mathcal{V}}_0 \widehat{\mathcal{V}}_\infty^{-1} = \widetilde{U}(0,\infty,w)\;, \quad \text{for } w \in H_{\pm}\;, \quad \text{i.e. for } \Im w \gtrless 0\;, \tag{3.38}$$

where the index $\pm$ refers to the discontinuity of $\widehat{\mathcal{V}}_\infty$ along the real w-line. The $U_{\pm}(w)$ are ($\mathbf{G}$-valued) holomorphic functions in H_+ and H_-, respectively, and related by

$$U_{\pm}(w) = \overline{U_{\mp}(\bar{w})}\;. \tag{3.39}$$

According to (3.24), further integrals of motion descend from the monodromy matrices (3.9) of $\widehat{\mathcal{V}}_0$ and $\widehat{\mathcal{V}}_\infty$. They may however be expressed in terms of the matrices $U_{\pm}(w)$: For real w it is

$$\begin{aligned}
\mathcal{M}_{\rm BM}(w) &= \mathcal{M}_0(w) \\
&\overset{(3.9)}{=} \lim_{\epsilon\to 0} \Big(\widehat{\mathcal{V}}_0(x,\gamma(w+{\rm i}\epsilon))\,\tau\Big(\widehat{\mathcal{V}}_0^{-1}\left(x,\gamma^{-1}(w+{\rm i}\epsilon)\right)\Big)\Big) \\
&\overset{\text{Fig.1}}{=} \lim_{\epsilon\to 0} \Big(\widehat{\mathcal{V}}_0(x,\gamma(w+{\rm i}\epsilon))\,\tau\Big(\widehat{\mathcal{V}}_0^{-1}\left(x,\gamma(w-{\rm i}\epsilon)\right)\Big)\Big) \\
&\overset{(3.38)}{=} \lim_{\epsilon\to 0} \Big(U_+(w)\,\widehat{\mathcal{V}}_\infty(x,\gamma(w+{\rm i}\epsilon))\,\tau\Big(\widehat{\mathcal{V}}_\infty(x,\gamma(w-{\rm i}\epsilon))U_-(w)\Big)^{-1}\Big) \\
&\overset{(3.10)}{=} U_+(w)\,\tau\big(U_-^{-1}(w)\big)
\end{aligned} \tag{3.40}$$

Throughout this calculation it is important that $x > |w+t|$. This ensures that the limits $x\to\infty$ and $\epsilon\to 0$ interchange as well as $\gamma(w+{\rm i}\epsilon) = \gamma^{-1}(w-{\rm i}\epsilon)$.

Vice versa, (3.40) can be understood as the essentially unique (Riemann-Hilbert) factorization of $\mathcal{M}_{\rm BM}$ into a product of matrices holomorphic in the upper and the lower half of the complex w-plane, respectively. The symmetry (3.13) of $\mathcal{M}_{\rm BM}$ further implies the relation

$$U_+(w)\,\tau\big(U_-^{-1}(w)\big) = U_-(w)\,\tau\big(U_+^{-1}(w)\big)\;. \tag{3.41}$$

Together with (3.39) this ensures reality of all matrix entries of $\mathcal{M}_{\rm BM}$ on the real w-axis:

$$\mathcal{M}_{\rm BM}(w) = \overline{\mathcal{M}_{\rm BM}(\bar{w})}\;. \tag{3.42}$$

The monodromy $\mathcal{M}_\infty$ associated to $\widehat{\mathcal{V}}_\infty$ follows from (3.38) and (3.40):

$$\mathcal{M}_\infty(w) = U_{\pm}^{-1}(w)\,U_{\mp}(w) \quad \text{for } w \in H_{\pm}\;. \tag{3.43}$$

Summarizing, we find that all the the integrals of motion identified according to the discussion in the previous section can be entirely expressed in terms of the $U_{\pm}(w)$. So far, we have however not answered the question of their physical content. For this purpose, we bring them into a more illustrative form. Starting from definitions (3.17), (3.38)

$$U_{\pm}(w) = \mathcal{V}(x\!=\!0,t)\;\mathcal{P}\exp\int_0^\infty dx\left(Q_1 + \frac{1+\gamma^2}{1-\gamma^2}\,P_1 - \frac{2\gamma}{1-\gamma^2}\,P_0\right)\;,$$

the t-independence may be exploited to calculate this expression for real w at the specific value $t=-w$ (assuming regularity of the currents):

$$U_{\pm}(w) = \mathcal{V}(x\!=\!0,t\!=\!-w)\;\mathcal{P}\exp\int_0^\infty dx\,\Big(Q_1(x,-w) \pm {\rm i}P_0(x,-w)\Big)\;. \tag{3.44}$$

The $\pm$-sign on the r.h.s. of (3.44) reflects the different limits $\lim_{\epsilon\to+0}\gamma(w\pm \mathrm{i}\epsilon)$. On the real w-axis $U_\pm(w)$ thus naturally factorizes into the product of a real and a compact part. The monodromy matrix $\mathcal{M}_{\rm BM}$ captures the real part of (3.44):

$$\begin{aligned}\mathcal{M}_{\rm BM}(w) &= \mathcal{V}(x\!=\!0,t\!=\!-w)\;\tau\big(\mathcal{V}^{-1}(x\!=\!0,t\!=\!-w)\big) \\ &\overset{(2.42)}{=} M(x\!=\!0,t\!=\!-w)\;, \qquad\qquad \text{for}\;\; w\in\mathbb{R}\;,\end{aligned} \tag{3.45}$$

whereas $\mathcal{M}_\infty(w)$ carries the compact part of (3.44).

Equation (3.45) provides a physical interpretation for the new integrals of motion. They comprise the values of the original field M on the symmetry axis $x=0$. Having been defined as spatially nonlocal charges for fixed t, they gain a definite localization in the two-dimensional spacetime at fixed x.[6]

Moreover, this shows that they contain the entire information about the solution. Together with the fact that $P_1(x=0)=0$, which follows from the equations of motion (2.49), the values on the symmetry axis $x\!=\!0$ allow to recover the field $\mathcal{V}$ everywhere. In some sense the initial values on a spacelike surface have been transformed into initial values along a timelike surface. Thus, the $U_\pm(w)$ build a complete set of constants of motion for this classical sector of solutions regular on the symmetry axis.

It remains to compute the Poisson algebra of the integrals of motion $U_\pm(w)$. According to their definition (3.38) we evaluate the general result (3.34) in the limit $x,x'\to 0$, $y,y'\to\infty$. The first four terms become

$$\left[\frac{\Omega_{\mathfrak{h}}}{v-w}\;,\;\overset{1}{U}(v)\,\overset{2}{U}(w)\right]\;,$$

for arbitrary indices $\pm$ at the U's.

The next two terms show the ambiguous behavior at coinciding endpoints. Depending on $x<x'$ or $x>x'$ they give the coefficient

$$f(x,v,w) \quad \text{or} \quad f(x',w,v)\;,$$

respectively, leaving to different results for different ways of taking the limit $x'\to x$. Here, the difference with the PCM becomes manifest: Since the spectral parameters depend on the spatial coordinates, in the limit $x,x'\to 0$ both $f(x,v,w)$ and $f(x',w,v)$ tend to 1 (cf. (3.5)). The sum

$$\lim_{x\to x'}\Big(\theta(x',x,y)f(x,v,w)+\theta(x,x',y)f(x',w,v)\Big)$$

thus is independent of how this limit is taken, keeping e.g. $x<x'$ or $x>x'$ or also $x=x'$ with $\theta(x,x,y)\equiv\frac{1}{2}$.

In a similar way, the ambiguity from the last two terms vanishes. In the limit $y,y'\to\infty$, the combinations $f(y,v,w)$ and $f(y',w,v)$ approach the same value. This common value is

[6] A similar relation holds for the monodromy matrix arising from timelike dimensional reduction (i.e. with a Euclidean two-dimensional world-sheet Σ) in the regular regions of the spacetime [13]. In that setting, singularities of the nonlocal charges in the spectral parameter plane are directly translated into singularities of the original fields in space-time.

however sensitive to the choice of indices $\pm$ at the U's, i.e. to the relative sign between the imaginary parts of v and w. If $\gamma(v)$ and $\gamma(w)$ lie in the same of the two regions D_+ and D_-, the function $f(y,v,w)$ tends to 1, whereas it tends to -1 otherwise (cf. (3.5)).

Thus, we arrive at the following Poisson algebra:

$$\left\{\overset{1}{U}_\pm(v)\,,\,\overset{2}{U}_\pm(w)\right\} \;=\; \left[\frac{\Omega_{\mathfrak{g}}}{v-w}\,,\,\overset{1}{U}_\pm(v)\,\overset{2}{U}_\pm(w)\right]\,, \tag{3.46}$$

$$\left\{\overset{1}{U}_\pm(v)\,,\,\overset{2}{U}_\mp(w)\right\} \;=\; \frac{\Omega_{\mathfrak{g}}}{v-w}\,\overset{1}{U}_\pm(v)\,\overset{2}{U}_\mp(w) - \overset{1}{U}_\pm(v)\,\overset{2}{U}_\mp(w)\frac{\Omega^\tau_{\mathfrak{g}}}{v-w}\,, \tag{3.47}$$

with $\Omega^\tau_{\mathfrak{g}} \equiv \Omega_{\mathfrak{h}} - \Omega_{\mathfrak{k}}$ obtained from $\Omega_{\mathfrak{g}}$ by applying the involution τ in one of the two spaces

$$\Omega_{\mathfrak{g}} = t^A \otimes t_A\,, \qquad \Omega^\tau_{\mathfrak{g}} = \tau(t^A)\otimes t_A = t^A \otimes \tau(t_A)\,.$$

Equations (3.46) build two semi-classical copies of the Yangian algebra that is well known from other $2d$ field theories [10, 11, 12]. By semi-classical we mean as usual that the Poisson brackets (3.46) coincide with the commutator of the $\hbar$-graded Yangian algebra in first order $\hbar$. The mixed relations (3.47) appear "twisted" by the involution τ with respect to those coming from the normal Yangian double.

Note that whereas (3.46) remains regular at coinciding arguments, (3.47) obviously becomes singular at $v\!=\!w$. However, since U_+ and U_- are defined in different domains, this singularity appears only in the limit on the real line and thus with a well-defined $\mathrm{i}\epsilon$-prescription. In other words, the Poisson algebra (3.46), (3.47) is compatible with the holomorphy properties of $U_\pm(w)$. For consistency, it may further be checked that the algebra (3.46), (3.47) is indeed compatible with the the restriction $U_\pm(w)\!\in\!\mathbf{G}$ and with the symmetry (3.41).

Remark 3.4 Let us recall the Poisson bracket (3.28) between the conformal factor σ and the integrals of motion $U_\pm(w)$ obtained above:

$$\{U_\pm(w), \sigma(x\!=\!\infty)\} \;=\; -\partial_w U_\pm(w)\,, \tag{3.48}$$

where we have assumed that the value of the conformal factor on the symmetry axis is fixed by the boundary conditions [3]. In the context of cylindrically symmetric $3d$ gravity coupled to scalar fields, the conformal factor $\exp\sigma$ at radial infinity has a well defined physical meaning. It contains the deficit angle describing the nontriviality of the asymptotically flat $3d$ metric and provides a measure of the total energy of the system. The simple form of its Poisson bracket with the new variables may have further consequences upon quantization [76].

Finally, we can also compute the symplectic structure on the Breitenlohner-Maison monodromy matrix $\mathcal{M}_{\rm BM}$, since we have identified this object within the canonical framework. It follows from (3.46), (3.47) and (3.40) that its matrix entries form the closed Poisson algebra:

$$\begin{aligned}\left\{\overset{1}{\mathcal{M}}_{\rm BM}(v)\,,\,\overset{2}{\mathcal{M}}_{\rm BM}(w)\right\} \;=\;& \\ &\frac{\Omega_{\mathfrak{g}}}{v-w}\,\overset{1}{\mathcal{M}}_{\rm BM}(v)\,\overset{2}{\mathcal{M}}_{\rm BM}(w) + \overset{1}{\mathcal{M}}_{\rm BM}(v)\,\overset{2}{\mathcal{M}}_{\rm BM}(w)\frac{\Omega_{\mathfrak{g}}}{v-w} \\ &- \overset{1}{\mathcal{M}}_{\rm BM}(v)\frac{\Omega^\tau_{\mathfrak{g}}}{v-w}\,\overset{2}{\mathcal{M}}_{\rm BM}(w) - \overset{2}{\mathcal{M}}_{\rm BM}(w)\frac{\Omega^\tau_{\mathfrak{g}}}{v-w}\,\overset{1}{\mathcal{M}}_{\rm BM}(v)\,.\end{aligned} \tag{3.49}$$

The singularity at $v\!=\!w$ is understood in the principal value sense.

Nonlocal charges for a timelike dilaton field

Here, we deal with the case of a globally timelike vector field $\partial_\mu\rho$, which allows to identify ρ with the time t. Accordingly, $\tilde\rho$ now describes the spatial coordinate x. The distinguished location $x=0$ which has played the role of the symmetry axis $r=0$ in the previous paragraph becomes now the origin $t=0$. With periodic spatial topology, this is the setting of the so-called cosmological Gowdy-models [49].[7] We will, however, just treat the asymptotic case $x\in\,]-\infty,\infty[$. The fundamental structures of the preceding section (the spacelike dilaton) reappear in this context from a somewhat different side. So, for this paragraph we fix

$$\rho=t\;,\quad \tilde\rho=x\in\,]-\infty,\infty[\;. \tag{3.50}$$

According to (3.24) the transition matrices again provide solutions of the linear system:

$$\begin{aligned}\widehat{\mathcal V}_{-\infty}(x,\gamma(w)) &\equiv \mathcal V(-\infty)\,U(-\infty,x,w)\;,\\ \widehat{\mathcal V}_{\infty}(x,\gamma(w)) &\equiv \mathcal V(\infty)\,U(\infty,x,w)\;.\end{aligned} \tag{3.51}$$

This time, the branch cut of Fig. 1 involved in the definition of the solutions (3.51) moves along the real w-axis without changing its length. Both these solutions turn out to be holomorphic inside of the unit disc $D_+\cup D_-$ in the γ-plane, thus in fact it is

$$\widehat{\mathcal V}_{\rm BM}=\widehat{\mathcal V}_{-\infty}=\widehat{\mathcal V}_{\infty}\;.$$

In particular, the objects

$$U(w)\equiv\widehat{\mathcal V}_{-\infty}(x,\gamma(w))\widehat{\mathcal V}_{\infty}^{-1}(x,\gamma(w))=I \tag{3.52}$$

superficially analogous to (3.38) are trivial here.

However, again we have identified $\widehat{\mathcal V}_{\rm BM}$ among the canonical objects. Its monodromy matrix $\mathcal M_{\rm BM}(w)$ for real w is given by

$$\mathcal M_{\rm BM}(w)=\lim_{\epsilon\to0}\left(\widehat{\mathcal V}_{\rm BM}(x,\gamma(w+{\rm i}\epsilon))\;\tau\Big(\widehat{\mathcal V}_{\rm BM}^{-1}(x,\gamma(w-{\rm i}\epsilon))\Big)\right)\;, \tag{3.53}$$

for $|w+x|<t$. Unlike (3.40) there is no way to express this matrix directly in terms of certain transition matrices. This is due to the fact that the limits $\epsilon\to0$ and $x\to\infty$ do not interchange in (3.53).

The matrix $\mathcal M_{\rm BM}(w)$ can be given more explicitly. Since $\mathcal M(w)$ is independent of x and t, we may evaluate it at $x=-w$ and in the limit $t\to0$. This yields:

$$\begin{aligned}\mathcal M_{\rm BM}(w) &= \lim_{\substack{t\to0\\ x=-w}}\left(\mathcal P\exp\int_{-\infty}^x dz\,L_1(z,\gamma)\quad\mathcal P\exp-\int_{-\infty}^x dz\;\tau\Big(L_1(z,\gamma)\Big)\right)\\ &= \mathcal V(x=-w,t=0)\;\tau\Big(\mathcal V^{-1}(x=-w,t=0)\Big)\\ &= M(x=-w,t=0)\;.\end{aligned} \tag{3.54}$$

Thus, $\mathcal M_{\rm BM}(w)$ again coincides with the values of the physical field M at $\rho=0$.

[7] See [53, 94] for a recent treatment of the Gowdy model in Ashtekar variables. The two Killing vector field reductions of pure Einstein gravity in terms of Ashtekar variables and the metric variables used here are equivalent [95, 120]. The explicit formulas of [120] allow to translate the results from one setting into the other.

After some calculation, the general result (3.34) further yields the Poisson algebra of the $\mathcal{M}_{\rm BM}$ here which turns out to coincide with (3.49):

$$\left\{ \overset{1}{\mathcal{M}}_{\rm BM}(v)\,,\,\overset{2}{\mathcal{M}}_{\rm BM}(w) \right\} = \tag{3.55}$$

$$\frac{\Omega_{\mathfrak{g}}}{v-w}\,\overset{1}{\mathcal{M}}_{\rm BM}(v)\,\overset{2}{\mathcal{M}}_{\rm BM}(w) + \overset{1}{\mathcal{M}}_{\rm BM}(v)\,\overset{2}{\mathcal{M}}_{\rm BM}(w)\frac{\Omega_{\mathfrak{g}}}{v-w}$$
$$-\,\overset{1}{\mathcal{M}}_{\rm BM}(v)\frac{\Omega_{\mathfrak{g}}^{\tau}}{v-w}\,\overset{2}{\mathcal{M}}_{\rm BM}(w) - \overset{2}{\mathcal{M}}_{\rm BM}(w)\frac{\Omega_{\mathfrak{g}}^{\tau}}{v-w}\,\overset{1}{\mathcal{M}}_{\rm BM}(v)\;.$$

This is by no means a consequence of (3.49), since the matrices $\mathcal{M}_{\rm BM}$ in both contexts descend from rather different definitions.

Via the Riemann-Hilbert decomposition of $\mathcal{M}_{\rm BM}$, discussed in (3.40), one can implicitly obtain the matrices $U_{\pm}$. They will satisfy the Poisson-structure (3.46), (3.47). Thus, together with (3.54) the final situation appears rather similar to the previous paragraph.

However, this result must be taken with some caution. Obviously, (3.54) looses its meaning if $M(x,t)$ diverges as $t\to 0$. Starting from arbitrary initial data at finite t, this divergence on the other hand is generic. What is actually described with (3.54) and (3.55) is the sector of the phase space where $M(x,t)$ behaves regularly at $t=0$. Note, that the canonical formulation obviously fails to cope with describing this truncated phase space: At $t=0$ the framework breaks down with the vanishing Lagrangian (2.45), whereas at finite t the condition of regularity at $t=0$ poses highly nontrivial implicit relations between the canonical coordinates and the momenta. Thus, the results of this paragraph should only be understood as an indication for some fundamental meaning of the Poisson algebra (3.49), (3.55) beyond the particular choice of Weyl coordinates (3.36).

Finally, let us mention another rather intriguing point of view for the coincidence of (3.49) and (3.55). Recall the setting of the spacelike dilaton (3.36) addressed above. In addition to the canonical (equal-time) symplectic structure, we could have derived an alternative Poisson structure with respect to the radius x.[8] The calculations of this paragraph show that these two Poisson structures of one model coincide for the values of the original fields on the symmetry axis $x=0$, i.e. for a complete set of observables. In this sense, these symplectic structures are essentially equivalent. It is tempting to speculate about further exploiting the fundamental structure (3.49) even in the case of a timelike dimensional reduction, i.e. the reduction to stationary axisymmetric spacetimes, where the canonical time is no longer present.

Summary

We have shown that the model (2.45) in Weyl coordinates (2.20) is completely described by a set of integrals of motion $U_{\pm}(w)$ defined as $\mathbf{G}$-valued functions which are holomorphic in the upper and the lower half of the complex plane, respectively. They are related by

$$U_{\pm}(w) = \overline{U_{\mp}(\bar{w})}\;, \tag{3.56}$$

[8] In a covariant theory this is a quite natural idea which has been discussed in particular to describe static settings [16]. For the Schwarzschild black hole e.g. one might doubt the distinct role of time in the canonical formalism since x and t change their character being space- and timelike, respectively, inside the horizon.

and subject to the condition

$$U_+(w)\,\tau\big(U_-^{-1}(w)\big) \;=\; U_-(w)\,\tau\big(U_+^{-1}(w)\big)\;. \tag{3.57}$$

The physical quantities are encoded in their matrix product

$$\mathcal{M}_{\rm BM}(w) = U_+(w)\tau\big(U_-^{-1}(w)\big)\;, \tag{3.58}$$

which according to (3.45) coincides with the original field $M(x,t)$ on the axis $\rho\!=\!0$:

$$\mathcal{M}_{\rm BM}(w) = M(\rho(x,t)\!=\!0\,,\;\tilde\rho(x,t)\!=\!-w)\;. \tag{3.59}$$

In particular, (3.56) and (3.57) imply that $\mathcal{M}_{\rm BM}(w)$ is a symmetric matrix with real matrix entries on the real w axis.

This structure has been revealed explicitly for the two definite choices of Weyl coordinates (3.36) and (3.50), i.e. having fixed the gauge freedom of conformal transformations. Since, according to (3.26), the $U_\pm(w)$ are invariant under conformal transformations, this structure extends also beyond these special choices. Its interplay with global properties of an arbitrary dilaton field ρ remains to be studied.

The Poisson algebra of the $U_\pm(w)$ is given by

$$\Big\{\overset{1}{U}_\pm(v)\,,\,\overset{2}{U}_\pm(w)\Big\} \;=\; \left[\frac{\Omega_{\mathfrak{g}}}{v-w}\,,\,\overset{1}{U}_\pm(v)\,\overset{2}{U}_\pm(w)\right]\;, \tag{3.60}$$

$$\Big\{\overset{1}{U}_\pm(v)\,,\,\overset{2}{U}_\mp(w)\Big\} \;=\; \frac{\Omega_{\mathfrak{g}}}{v-w}\,\overset{1}{U}_\pm(v)\,\overset{2}{U}_\mp(w) - \overset{1}{U}_\pm(v)\,\overset{2}{U}_\mp(w)\frac{\Omega_{\mathfrak{g}}^\tau}{v-w}\;. \tag{3.61}$$

It gives rise to a closed Poisson algebra of the matrix entries of $\mathcal{M}_{\rm BM}$:

$$\Big\{\overset{1}{\mathcal{M}}_{\rm BM}(v)\,,\,\overset{2}{\mathcal{M}}_{\rm BM}(w)\Big\} \;= \tag{3.62}$$

$$\frac{\Omega_{\mathfrak{g}}}{v-w}\,\overset{1}{\mathcal{M}}_{\rm BM}(v)\,\overset{2}{\mathcal{M}}_{\rm BM}(w) + \overset{1}{\mathcal{M}}_{\rm BM}(v)\,\overset{2}{\mathcal{M}}_{\rm BM}(w)\frac{\Omega_{\mathfrak{g}}}{v-w}$$
$$-\,\overset{1}{\mathcal{M}}_{\rm BM}(v)\frac{\Omega_{\mathfrak{g}}^\tau}{v-w}\,\overset{2}{\mathcal{M}}_{\rm BM}(w) - \overset{2}{\mathcal{M}}_{\rm BM}(w)\frac{\Omega_{\mathfrak{g}}^\tau}{v-w}\,\overset{1}{\mathcal{M}}_{\rm BM}(v)\;.$$

Remark 3.5 Upon formal expansion around $w=\infty$, the Poisson algebra (3.60) coincides with the semi-classical Yangian structure which was introduced by Drinfeld [27] in the framework of Hopf algebras. To describe the Yangian double [28, 82, 11] it is usually convenient to take two copies of (3.60) with formal expansions around $w=0$ and $w=\infty$, respectively. In (3.60), (3.61) in contrast, the $U_\pm(w)$ do not arise as formal power series but as definite functions allowing holomorphic expansion in the upper and the lower half of the complex plane, respectively. The formal expansions around $w\!=\!0$ and $w\!=\!\infty$ hence are no appropriate parametrization.

3.4 Symmetries: The Geroch group revisited

With the integrals of motion $U_\pm(w)$ identified in the previous section, one can study the symmetries which they generate via their adjoint action in the canonical Poisson structure. As it turns out [75, 77], this yields a canonical realization of the Geroch group [47] with the underlying Yangian algebra (3.60), (3.61). The transformations which close into an affine algebra (the loop algebra $\hat{\mathfrak{g}}$, cf. (2.35)) do not preserve the symplectic structure. This is a particular example of the Lie-Poisson action of dressing groups generated by the transition matrices of integrable models [113, 6, 84]. For the integrable models studied so far within the framework of the quantum inverse scattering method, the integrals of motion are encoded in the eigenvalues of the transition matrices. Here, in contrast, the transition matrices $U_\pm(w)$ themselves are conserved charges.

The Geroch group and the linear system

In this paragraph, we sketch how the action of the Geroch group may be encoded in an action on the linear system (3.1). Since our main goal is the canonical realization of the Geroch group in the next paragraph, we keep the discussion rather brief, referring to [61, 13, 99] for details.

We have seen the one to one correspondence between solutions $\mathcal{V}$ of the original equations of motion (2.49) and the associated solutions $\widehat{\mathcal{V}}_{\rm BM}$ of the linear system (3.1). The latter allow the factorization (3.15)

$$\widehat{\mathcal{V}}_{\rm BM}(\gamma) = S_{\rm BM}(w)\widehat{\mathcal{V}}_{\rm BZ}(\gamma)\ , \tag{3.63}$$

into a matrix $S_{\rm BM}(w)$ living in the w-plane and a matrix $\widehat{\mathcal{V}}_{\rm BZ}(\gamma)$ living in the γ-plane and invariant under the involution τ^∞ from (3.8). Conversely, this equation shows how to obtain $\widehat{\mathcal{V}}_{\rm BM}(\gamma)$ from $\mathcal{M}_{\rm BM}$: Decompose $\mathcal{M}_{\rm BM}$ according to (3.14) and determine the unique $\widehat{\mathcal{V}}_{\rm BZ}(\gamma)$ invariant under τ^∞, such that the product (3.63) as a function of γ is holomorphic inside the unit disc. Thus, one obtains $\widehat{\mathcal{V}}_{\rm BM}(\gamma)$, which in particular is sufficient to reproduce the original fields $\mathcal{V}$ according to (3.11).

This procedure describes the finite transformations of the Geroch group, which generate an arbitrary solution $\widehat{\mathcal{V}}_{\rm BM}$ from the vacuum solution $\widehat{\mathcal{V}}^0_{\rm BM} = I$. They are parametrized by $\mathbf{G}$-valued matrices $S(w)$. The group structure is simply given by matrix multiplication: On a given solution $\widehat{\mathcal{V}}_{\rm BM}$, $S(w)$ acts by left multiplication which in turn induces a right multiplication to restore the holomorphy inside the unit γ-disc. The monodromy matrix $\mathcal{M}_{\rm BM}$ transforms as

$$\mathcal{M}_{\rm BM}(w) \mapsto S(w)\mathcal{M}_{\rm BM}(w)\,\tau\big(S^{-1}(w)\big)\ . \tag{3.64}$$

On the algebra level, this action takes the following form: Parametrizing the algebra action by a $\mathfrak{g}$-valued meromorphic function $\Lambda(w)$ we define

$$\delta_\Lambda \widehat{\mathcal{V}}_{\rm BM}(\gamma) \equiv \Lambda(w)\widehat{\mathcal{V}}_{\rm BM}(\gamma) + \widehat{\mathcal{V}}_{\rm BM}(\gamma)\Upsilon_\Lambda(\gamma)\ , \tag{3.65}$$

where $\Upsilon_\Lambda(\gamma)$ is the unique function invariant under τ^∞ which restores the holomorphy of $\delta_\Lambda\widehat{\mathcal{V}}_{\rm BM}(\gamma)$ inside the unit disc. The infinitesimal version of (3.64) accordingly reads

$$\delta_\Lambda \mathcal{M}_{\rm BM}(w) = \Lambda(w)\mathcal{M}_{\rm BM}(w) - \mathcal{M}_{\rm BM}(w)\,\tau(\Lambda(w))\ . \tag{3.66}$$

We have now associated a finite transformation of the Geroch group to each element of the phase space, by which it is generated from the vacuum solution. According to (3.63) the Geroch group is generated by meromorphic functions $S(w)$ mapping the complex w-plane into the group $\mathbf{G}$. Denote this group by $\mathbf{G}^\infty$. The phase space may be understood as an infinite-dimensional coset space

$$\mathbf{G}^\infty/\mathbf{H}^\infty \;, \tag{3.67}$$

where $\mathbf{H}^\infty$ refers to the subgroup of $\mathbf{G}$-valued functions on the γ-plane invariant under τ^∞.[9] This subgroup describes the freedom of right multiplication of $\widehat{\mathcal{V}}(\gamma)$ which leaves the associated monodromy matrix (3.9) invariant.

The particular elements $\widehat{\mathcal{V}}_{\rm BM}$ may be viewed as a certain representative system of this coset space (3.67). Their $\mathbf{H}^\infty$ gauge freedom is fixed (3.63) by demanding holomorphy inside of the unit disc in the γ-plane. This is a generalization of the triangular gauge discussed for the finite-dimensional coset space $\mathbf{G}/\mathbf{H}$ in (2.25). The action of the Geroch group as described above is the action of the coset space (3.67) on itself. In analogy to (2.14) the linearized action on $\mathbf{G}^\infty$ becomes highly nonlinear on the fixed representation system $\widehat{\mathcal{V}}_{\rm BM}$ of the coset space. On the algebra level, the action of the symmetry (3.65) is parametrized by $\Lambda\in\mathfrak{g}^\infty$, while $\Upsilon_\Lambda\in\mathfrak{h}^\infty$ is required to restore the generalized triangular gauge.

Let us finally recover the structure of the Geroch group that we have encountered earlier in the model of pure Einstein gravity. There, the Geroch group has been described as the affine algebra $\widehat{\mathfrak{g}}$ (2.35) with the action of the generators given in (2.14) and (2.33), (2.34). The algebra $\mathfrak{g}^\infty$ of meromorphic $\mathfrak{g}$-valued functions is formally related to $\widehat{\mathfrak{g}}$ by Laurent expansion around a given point w_0.

With $w_0=\infty$ the (truncated) Laurent expansion

$$\Lambda(w) \;=\; \Lambda_0 + w\Lambda_1 + w^2\Lambda_2 + \dots \;, \tag{3.68}$$

yields one half of the affine algebra. Since these $\Lambda(w)$ introduce a singularity at $\gamma=0$ they require a compensating transformation Υ_Λ according to (3.65) which acts nontrivially on the physical fields eventually obtained from (3.11). The expansion (3.68) leads to explicit recurrence relations for this action [99]. A closer check of (3.65) shows that indeed the parameter Λ_1 describes the action (2.14) of the zero modes $\mathfrak{g}\otimes z^0$, whereas Λ_2 corresponds to the action (2.33), (2.34) of the elements $\mathfrak{g}\otimes z$ in $\widehat{\mathfrak{g}}$. Thus, (3.65) generalizes the action (2.14) of the zero modes of (2.35) to that half of the affine algebra which acts nontrivially on the physical fields (cf. Remark 2.2). The other half of the affine algebra may be associated with the Taylor expansion of $\Lambda(w)$ around $w_0=\infty$ [61, 13, 99].

The canonical realization of the Geroch group

Here, we present the relation of the Geroch group described in the previous paragraph with the integrals of motion $U_\pm(w)$ that we have obtained in section 3.3. It turns out that this provides a natural realization of the Geroch group via the canonical Poisson structure. For definiteness, we assume the Weyl gauge (3.36) whenever necessary, such that, in particular,

[9] To properly define $\mathbf{H}^\infty$ as a subgroup of $\mathbf{G}^\infty$ one should regard (3.67) for fixed values of x and t, with γ and w related by (3.3). See [13, 63] for the mathematical details.

the solution $\widehat{\mathcal{V}}_{\rm BM}$ of the linear system is given by (3.37). As has been discussed above, the entire symmetry structure also survives relaxing of this gauge choice.

Recall that the $U_\pm(w)$ live in a matrix representation of $\mathbf{G}$, in particular each matrix entry thus represents an element of $C(\mathbf{G})$. Define in this representation the matrix valued operator

$$G_\pm(w) \;\equiv\; {\rm ad}_{U_\pm(w)} U_\pm^{-1}(w) \;, \tag{3.69}$$

where "${\rm ad}$" denotes the adjoint action via the canonical Poisson structure. To be precise, the action of $G_\pm(w)$ on any phase space function f is defined as

$$G_\pm^{ab}(w)\, f \;\equiv\; \left\{ U_\pm^{am}(w), f \right\} \left(U_\pm^{-1}\right)^{mb}(w) \;,$$

in matrix indices a, b. Since the $U_\pm(w)$ are integrals of motion, this action is a symmetry of the equations of motion of the theory. It is illustrative to calculate the transformation behavior of the monodromy matrix $\mathcal{M}_{\rm BM}$ according to (3.58) and (3.60), (3.61):

$$\overset{1}{G}_\pm(v)\; \overset{2}{\mathcal{M}}_{\rm BM}(w) \;=\; \frac{\Omega_{\mathfrak{g}}}{v-w}\; \overset{2}{\mathcal{M}}_{\rm BM}(w) - \overset{2}{\mathcal{M}}_{\rm BM}(w)\, \frac{\Omega_{\mathfrak{g}}^{\tau}}{v-w} \;. \tag{3.70}$$

This motivates the definition of the following symmetry operator

$$G[\Lambda] \;\equiv\; {\rm tr} \left(\int_{\ell_+} \frac{dv}{2\pi {\rm i}}\, \Lambda(v)\, G_+(v) \;+\; \int_{\ell_-} \frac{dv}{2\pi {\rm i}}\, \Lambda(v)\, G_-(v) \right) \;, \tag{3.71}$$

for any algebra-valued function $\Lambda(w) \in \mathfrak{g}$, regular along the real w-axis and vanishing as $w \to \infty$, where the path $\ell = \ell_+ \cup \ell_-$ is chosen to encircle the real w-axis, such that $\ell_\pm \in H_\pm$ and $\Lambda(w)$ is holomorphic inside the enclosed area. Then, we obtain from (3.70)

$$G[\Lambda]\, \mathcal{M}_{\rm BM}(w) \;=\; \Lambda(w) \mathcal{M}_{\rm BM}(w) - \mathcal{M}_{\rm BM}(w)\, \tau(\Lambda(w)) \;, \tag{3.72}$$

which coincides with (3.66). This already reproduces the infinitesimal action of the Geroch group in the canonical framework. Moreover, (3.72) shows that the symmetry group (3.71) acts transitively among solutions which behave analytically on the symmetry axis $\rho = 0$ (cf. (3.59)).

Let us check, if we can also recover the action (3.65) on the solution $\widehat{\mathcal{V}}_{\rm BM}$ of the linear system. Evaluating the key formula (3.34) according to the definitions (3.71) and (3.37) leads to:

$$G[\Lambda]\, \widehat{\mathcal{V}}_{\rm BM}(x,t,\gamma(w)) \;=\; \Lambda(w) \widehat{\mathcal{V}}_{\rm BM}(x,t,\gamma(w)) - \widehat{\mathcal{V}}_{\rm BM}(x,t,\gamma(w))\, \Upsilon_\Lambda(x,t,\gamma(w)) \;,$$

where

$$\begin{aligned} \Upsilon_\Lambda(x,t,\gamma(w)) \;=&\; \int_\ell \frac{dv}{2\pi {\rm i}(v-w)} \left[\widehat{\mathcal{V}}_{\rm BM}^{-1}\, \Lambda\, \widehat{\mathcal{V}}_{\rm BM} \right]_{\mathfrak{h}} \\ &+ \frac{1-\gamma^2(w)}{\gamma(w)} \int_\ell \frac{dv}{2\pi {\rm i}(v-w)}\, \frac{\gamma(v)}{1-\gamma^2(v)} \left[\widehat{\mathcal{V}}_{\rm BM}^{-1}\, \Lambda\, \widehat{\mathcal{V}}_{\rm BM} \right]_{\mathfrak{k}} \;, \end{aligned} \tag{3.73}$$

with the algebra projections $[.]_{\mathfrak{h}}$, $[.]_{\mathfrak{k}}$ corresponding to the decomposition (2.37). The matrix $\Lambda(w)$ depends on the constant spectral parameter w; in contrast, $\Upsilon_\Lambda(x,t,\gamma(w))$ depends on the variable spectral parameter γ and obviously satisfies

$$\Upsilon_\Lambda(x,t,\gamma(w)) \;=\; \tau^\infty(\Upsilon_\Lambda(x,t,\gamma(w))) \;=\; \tau\big(\Upsilon_\Lambda(x,t,\gamma^{-1}(w))\big) \;.$$

Thus, we find agreement with (3.65) and have in particular obtained a closed expression for the compensating $\mathfrak{h}^\infty$–rotation $\Upsilon_\Lambda(\gamma)$. Indeed, it follows from the form of Υ_Λ (3.73), that the right multiplication of $\widehat{\mathcal{V}}_{\rm BM}$ with Υ_Λ removes all singularities caused by the left multiplication with $\Lambda(w)$ from the unit disc (note that the path ℓ surrounds the unit disc in the γ-plane).

With the symmetry operator (3.71) at hand, we can directly calculate the infinitesimal action of the Geroch group on all the original fields of the model. According to the general formula (3.29) it follows that:

$$\begin{aligned} G[\Lambda]\, \mathcal{V}(x) &= \int_\ell \frac{dv}{2\pi \mathrm{i}} \left(\frac{2\gamma}{\rho(1-\gamma^2)}\, \mathcal{V}(x) \left[\widehat{\mathcal{V}}_{\rm BM}^{-1} \Lambda \widehat{\mathcal{V}}_{\rm BM}\right]_{\mathfrak{k}} \right) \\ &= -\int_{\gamma(\ell)} \frac{d\gamma}{2\pi \mathrm{i}\, \gamma} \left(\mathcal{V}(x) \left[\widehat{\mathcal{V}}_{\rm BM}^{-1} \Lambda \widehat{\mathcal{V}}_{\rm BM}\right]_{\mathfrak{k}} \right) . \end{aligned} \tag{3.74}$$

The currents $P_\pm = \frac{1}{2}(P_0 \pm P_1)$ transform as:

$$\begin{aligned} G[\Lambda]\, P_\pm(x) &= \int_\ell \frac{dv}{2\pi \mathrm{i}} \left[\frac{2\gamma}{\rho(1\pm\gamma)^2} \left[\widehat{\mathcal{V}}_{\rm BM}^{-1} \Lambda \widehat{\mathcal{V}}_{\rm BM}\right]_{\mathfrak{h}},\, P_\pm(x) \right] \\ &\qquad \mp \int_\ell \frac{dv}{2\pi \mathrm{i}}\, \frac{4\gamma^2 \partial_\pm \rho}{\rho^2 (1\pm\gamma)^2 (1-\gamma^2)} \left[\widehat{\mathcal{V}}_{\rm BM}^{-1} \Lambda \widehat{\mathcal{V}}_{\rm BM}\right]_{\mathfrak{k}} . \end{aligned} \tag{3.75}$$

Equivalent forms of these infinitesimal symmetry transformations of the Geroch group have been stated in [51, 119, 99].

The symmetry action on the conformal factor σ is given by

$$G[\Lambda]\, \sigma \;=\; \int_\ell \frac{dv}{2\mathrm{i}\pi}\, \mathrm{tr} \left(\Lambda \partial_v \widehat{\mathcal{V}}_{\rm BM} \widehat{\mathcal{V}}_{\rm BM}^{-1} \right) , \tag{3.76}$$

in accordance with the formula derived in [99]. Formula (3.76) is easily obtained from (3.27).

The algebraic structure of the symmetry operators (3.71) is most conveniently obtained from (3.72), which immediately gives rise to

$$\Big[G[\Lambda_1], G[\Lambda_2] \Big] \;=\; G\Big[[\Lambda_1, \Lambda_2] \Big] . \tag{3.77}$$

Like in the previous paragraph the symmetry algebra is parametrized by meromorphic $\mathfrak{g}$-valued functions. Half of the affine algebra (2.35) may again be recovered by formal Laurent expansion around $w = \infty$.

Recovering the affine algebra

Definition (3.69) together with (3.60) yields

$$\left[\overset{1}{G}_\pm(v) , \overset{2}{G}_\pm(w) \right] \;=\; \frac{1}{v-w} \left[\Omega_{\mathfrak{g}} ,\, \overset{1}{G}_\pm(v) + \overset{2}{G}_\pm(w) \right] , \tag{3.78}$$

The commutator on the r.h.s. encodes the half of an affine algebra in its Taylor expansion around $v = \infty$, $w = \infty$ [39]:

$$G_\pm(w) \;=\; I + \frac{1}{w} G_{1\,\pm} + \frac{1}{w^2} G_{2\,\pm} + \ldots \;,$$

which corresponds to the expansion (3.68) in the sense that:

$$G[\Lambda_n w^n] \;=\; \tfrac{1}{2}(G_{n\,+}+G_{n\,-})\;.$$

This relation follows from evaluating (3.71). There is a slight subtlety here, since strictly speaking the functions $\Lambda(w)=\Lambda_n w^n$ do not belong to the class of functions for which we have defined (3.71). As the integrand is singular at infinity, definition (3.71) depends on the precise choice of the contour in this region, which has not been specified above. Expansion of (3.74) around $w=\infty$ yields the action of these components on the physical fields. With the asymptotic behavior

$$\gamma(w) \;=\; \frac{1}{2w}\,\rho - \frac{1}{2w^2}\,\rho\tilde{\rho} + \ldots \tag{3.79}$$

it is possible to expand $\widehat{\mathcal{V}}_{\rm BM}$ according to

$$\widehat{\mathcal{V}}_{\rm BM}(x,t,\gamma) \;=\; \mathcal{V}\Big(I + \frac{1}{w}\,\mathcal{V}_1 + \frac{1}{w^2}\,\mathcal{V}_2 + \ldots\Big)\;, \tag{3.80}$$

with

$$\mathcal{V}_1 = \int_0^x dy\;\rho(y)\mathcal{V}(y)P_0(y)\mathcal{V}^{-1}(y)\;.$$

Then, (3.74) yields the following action (up to gauge transformations (2.38)):

$$\begin{aligned} G[\Lambda_1 w]\,\mathcal{V} &\;=\; \Lambda_1\mathcal{V}\;, \\ G[\Lambda_2 w^2]\,\mathcal{V} &\;=\; \left[\Lambda_2\,,\mathcal{V}\mathcal{V}_1\mathcal{V}^{-1}\right]\mathcal{V}\;, \\ G[\Lambda_2 w^2]\,P_\pm &\;=\; \mp\left[\rho\mathcal{V}^{-1}\Lambda_2\mathcal{V}\,,\,P_\pm\right] \mp \partial_\pm\rho\left[\mathcal{V}^{-1}(x)\Lambda_2\mathcal{V}(x)\right]_{\mathfrak{k}}\;. \end{aligned} \tag{3.81}$$

This coincides with the structure found in (2.31), (2.33) and (2.34) for $\mathfrak{g}=\mathfrak{sl}(2)$. In particular, it may easily be checked, that in this case the matrix $\mathcal{V}\mathcal{V}_1\mathcal{V}^{-1}$ indeed covers the first dual potentials (2.26) and (2.32).

The associated affine charges may be obtained from a formal expansion of the linear system (3.1) in the following way: Interpreting (3.1) as a formal power series in w^{-1}, the particular transition matrix $\widehat{\mathcal{V}}_{\rm BM}(x,t,\gamma(w))$ from (3.37) admits an expansion according to (3.80). Performing the limit $x\to\infty$ in each of the coefficients separately leads to a series

$$U(w) \;\equiv\; I + \frac{1}{w}\,U_1 + \frac{1}{w^2}\,U_2 + \ldots\;, \qquad \text{with}\;\; U_n \equiv \lim_{x\to\infty}\mathcal{V}_n\;. \tag{3.82}$$

The first two charges obtained this way are

$$\begin{aligned} U_1 &\;=\; \int_0^\infty dx\;\rho(x)\mathcal{V}(x)P_0(x)\mathcal{V}^{-1}(x)\;, \\ U_2 &\;=\; \tfrac{1}{2}\,U_1^2 \;+\; \tfrac{1}{2}\int_0^\infty dx\int_x^\infty dy\;\rho(x)\rho(y)\Big[\mathcal{V}(x)P_0(x)\mathcal{V}^{-1}(x)\,,\,\mathcal{V}(y)P_0(y)\mathcal{V}^{-1}(y)\Big] \\ &\qquad + \tfrac{1}{2}\int_0^\infty dx\;\rho^2(x)\mathcal{V}(x)P_1(x)\mathcal{V}^{-1}(x) + \int_0^\infty dx\;\rho(x)\tilde{\rho}(x)\mathcal{V}(x)P_0(x)\mathcal{V}^{-1}(x)\;. \end{aligned} \tag{3.83}$$

It may be checked, that they generate the action (3.81). However, it is important to notice that in general there is no relation between the formal power series $U(w)$ defined in (3.82) and the integrals of motion $U_\pm(w)$ from (3.38). This is due to the fact, that the limits $w \to \infty$ and $x \to \infty$ do not interchange (manifest in the breakdown of the expansion (3.79) at $w = -t + x$). In particular, all the U_n are real, whereas the $U_\pm(w)$ are complex with (3.56). Nevertheless, the formal series $U(w)$ generates the same action as the operators (3.71) defined via the $U_\pm(w)$.

Remark 3.6 The closed expressions (3.74), (3.75) of the symmetry action on the physical fields contain the pivotal term $\left[\widehat{\mathcal{V}}_{\rm BM}^{-1} \Lambda \widehat{\mathcal{V}}_{\rm BM}\right]$ which is hardly computable explicitly. The affine expansion (3.68) of the symmetry group has the seeming advantage, that it allows for explicit expressions of the associated charges (3.83) and the action of the symmetry (3.81). However, to obtain infinitesimal transformations which are integrated to physically meaningful solutions, the entire formal power series in (3.68) has to be summed up, i.e. the same amount of work is required. The closed form of (3.71) captures the structure of the full symmetry group. In particular, it provides precise control over the deviation of this action from a symplectic one (cf. (3.90) below) which later on becomes essential for the purpose of quantization.

Remark 3.7 We have given the canonical realization of the symmetry algebra (3.65). According to the discussion above this may formally be embedded into that half of the affine algebra $\widehat{g}$ (2.35) which acts nontrivially on the physical fields. There is no canonical realization of the other half and the central extension k for the following reason: According to Remark 2.2, the other half of the Geroch group leaves the physical fields $\mathcal{V}$ invariant and acts by shifting the dual potentials encoded in a solution $\widehat{\mathcal{V}}$ of the linear system (3.1). However, to set up the canonical framework we had to identify the particular solution $\widehat{\mathcal{V}}_{\rm BM}$ as a unique functional of the physical fields $\mathcal{V}$, which e.g. enabled us to obtain the symplectic structure (3.62). There is hence no canonical object which would transform nontrivially while the canonical fields are left invariant.

In other words, to incorporate the other half of the affine algebra and the central extension of (2.35), the phase space would have to be enlarged by additional gauge degrees of freedom (corresponding to $\mathfrak{h}^\infty$ in (3.67)). So far, it is not clear how to achieve this canonically, say, on the Lagrangian level. See [63, 102] for further discussion.

Lie-Poisson actions

Definition (3.71) implies that the action of the Geroch group is not symplectic. Rather, this type of operator generates a Lie-Poisson action, i.e. an action which does not preserve the Poisson structure on the phase space but on the direct product of the phase space with the symmetry group. In this paragraph, we briefly recall the mathematical concept of Lie-Poisson actions and show how the canonical realization of the Geroch group matches this framework. For the details and proofs we refer to [6, 84].

The action of a Lie group G on a symplectic manifold M is a map

$$G \times M \to M;\; g \times m \mapsto gm\;. \tag{3.84}$$

It naturally induces a map $C(M) \to C(M)$ by $f \mapsto f \circ g\,;\; f \circ g\,(m) \equiv f(gm)$. The action (3.84) is called symplectic, if for fixed $g \in G$ it is a Poisson map, i.e. it is compatible with the symplectic structure on M:

$$\{f_1 \circ g, f_2 \circ g\} = \{f_1, f_2\} \circ g\;,$$

for any two functions $f_1, f_2 \in C(M)$. The infinitesimal version of this condition reads

$$\{Xf_1, f_2\} + \{f_1, Xf_2\} = X\{f_1, f_2\} \ , \tag{3.85}$$

where X is the vector field related to the action of $g \in G$ and may be understood as an element of the associated Lie algebra $\mathfrak{g}$. Every infinitesimal action of this kind is locally generated by a charge

$$Xf_1 = \{Q, f_1\} \ , \tag{3.86}$$

and vice versa every action generated as (3.86) is obviously symplectic (due to the Jacobi-identities). An example of a symplectic action in our model is e.g. given by the action of the zero modes of the affine algebra (2.14), which is generated by the charges U_1 from (3.83).

For the subsequent generalization it is convenient to also state the dual version of (3.85). The action $f \mapsto Xf$ induces the dual map

$$\xi : C(M) \to C(M) \otimes \mathfrak{g}^*; \quad f \mapsto \xi_f \in C(M) \otimes \mathfrak{g}^*; \quad \xi_f(X \in \mathfrak{g}) \equiv Xf \ ,$$

in terms of which a symplectic action (3.85) satisfies:

$$\{\xi_{f_1}, f_2\} + \{f_1, \xi_{f_2}\} = \xi_{\{f_1, f_2\}} \ . \tag{3.87}$$

Let the group G now be a Lie-Poisson group, i.e. equipped with a symplectic structure

$$C(G) \times C(G) \to C(G) \ , \tag{3.88}$$

such that the group multiplication is a Poisson map. The Poisson structure naturally induces a Lie-algebra structure on $\mathfrak{g}^*$ (loosely speaking obtained from the differential of (3.88)). The space $G \times M$ then is a symplectic space with the product symplectic structure:

$$\{f_1, f_2\}_{G\times M}(g, m) \ = \ \{f_1(\cdot, m), f_2(\cdot, m)\}_G(g) + \{f_1(g, \cdot), f_2(g, \cdot)\}_M(m) \ . \tag{3.89}$$

To evaluate the r.h.s. the functions f_i are understood as functions on G with parameter m and as functions on M with parameter g, respectively. The action of a Lie-Poisson group on a symplectic manifold M is called a Lie-Poisson action, if (3.84) is a Poisson map, where $G \times M$ is equipped with (3.89). Compared with (3.87), the infinitesimal form of a Lie-Poisson action gets an additional contribution:

$$\{\xi_{f_1}, f_2\} + \{f_1, \xi_{f_2}\} = \xi_{\{f_1, f_2\}} - [\xi_{f_1}, \xi_{f_2}] \ . \tag{3.90}$$

The commutator on the r.h.s. refers to the Lie-bracket induced on $\mathfrak{g}^*$. This explicitly shows that a nonabelian Lie-Poisson action is not symplectic.

In our model the action of the generators $G_\pm(w)$ is precisely of the form (3.90). Evaluating definition (3.69) yields

$$\{G_\pm(w)f_1, f_2\} + \{f_1, G_\pm(w)f_2\} = G_\pm(w)\,\{f_1, f_2\} - [G_\pm(w)f_1\,, G_\pm(w)f_2] \ ,$$

where the commutator is understood for the matrix-valued action of $G_\pm(w)$. This coincides with (3.90). In fact every Lie-Poisson action is at least locally generated by an operator of the form (3.69) [6, 84]; this is the nonabelian generalization of (3.86).

In particular, dressing transformations in normal integrable systems are generated by an operator (3.69) where $U(w)$ denotes the transition matrix of the Lax connection, the eigenvalues of which give charges in involution. In our model in contrast the matrices $U_\pm(w)$ are integrals of motion themselves and parametrize the entire phase space. Rather than constructing the Lie-Poisson action (3.71) we could alternatively consider the pure symplectic action of the matrix entries of the $U_\pm(w)$ via Poisson brackets. However, though this action is certainly symplectic, it allows neither explicit exponentiation nor a closed form of the commutator algebra, in contrast to (3.72) and (3.77).

3.5 The Abelian sector

The results of this chapter simplify significantly if the group **G** is Abelian. In this case, all equations linearize and allow an explicit solution. Thus, truncating the model to its Abelian subsector may serve as a simple illustration or may be viewed as a testing ground for issues like implementing further symmetries or approaching the quantization of the model.

Here, we illustrate this for $\mathbf{G}=U(1)$. In the context of four-dimensional Einstein gravity (2.21) this corresponds to a diagonal matrix M_{ab}, i.e. cylindrical gravitational waves restricted to collinear polarization. These solutions have already been discovered by Einstein and Rosen [33]. Quantization of this sector has been studied as a midi-superspace model of quantum gravity [79, 3, 4]. With Euclidean signature of the two-dimensional world-sheet, this truncation is the one from stationary to static solutions of Einstein's equations.

Like in (2.21) we choose Weyl coordinates (3.36), identifying the dilaton ρ with the radius x. Parametrize M by

$$M \equiv \begin{pmatrix} e^{\varphi} & 0 \\ 0 & e^{-\varphi} \end{pmatrix} .$$

The Ernst equation (2.22) in this case reduces to the cylindrical wave equation

$$-\partial_t^2\varphi + x^{-1}\partial_x\varphi + \partial_x^2\varphi \;=\; 0 \;, \tag{3.91}$$

with general solution

$$\varphi(x,t) \;=\; \int_0^\infty dk\Big(A_+(k)\,J_0(kx)e^{\mathrm{i}kt} + A_-(k)\,J_0(kx)e^{-\mathrm{i}kt}\Big) \;, \tag{3.92}$$

where J_0 denote Bessel functions of the first kind. Another representation of the general solution of (3.91) is given by

$$\varphi(x,t) \;=\; \oint_\ell \frac{dv}{2\pi\mathrm{i}}\, m(v)\,\gamma^{-1}\partial_v\gamma(v) \;=\; -\oint_\ell \frac{dv}{2\pi\mathrm{i}}\, \partial_v m(v)\ln(\gamma(v)) \;, \tag{3.93}$$

with the spectral parameter γ from (3.3) and a path ℓ encircling the moving branch cut in the w-plane (cf. Figure 1). This representation even allows for an explicit solution of the linear system (3.1):

$$\hat\nu(\gamma(w)) \;=\; \oint_\ell \frac{dv}{2\pi\mathrm{i}}\, \frac{m(v)}{\gamma(v)-\gamma(w)}\, \partial_v\gamma(v) \;, \tag{3.94}$$

where $\hat{\nu}$ is one diagonal component of $\ln \widehat{\mathcal{V}}$.

The general solution of (3.91) is thus parametrized by a real function $m(w)$ or by the complex functions

$$A_+(k) = \overline{A_-(k)} \quad \text{with} \quad k > 0 \; .$$

Let us illustrate their relation. On the axis $x\!=\!0$ it is

$$\varphi(x\!=\!0,t) \;=\; \int_0^\infty dk\, A_+(k) e^{\mathrm{i}kt} \;+\; \int_0^\infty dk\, A_-(k) e^{-\mathrm{i}kt} \;=\; m(w\!=\!-t)\; .$$

This is nothing but the decomposition of a function on the real line into the sum of two functions holomorphic in the upper and the lower half of the complex plane, respectively. Comparing this decomposition to the nonabelian case (3.40), (3.45) we see the embedding of the abelian case according to

$$\begin{aligned} \mathcal{M}_{\mathrm{BM}}(w) &= \exp \begin{pmatrix} m(w) & 0 \\ 0 & -m(w) \end{pmatrix} , \\ U_\pm(w) &= \exp \begin{pmatrix} \int_0^\infty dk\, A_\pm(k) e^{\pm \mathrm{i}kt} & 0 \\ 0 & -\int_0^\infty dk\, A_\pm(k) e^{\pm \mathrm{i}kt} \end{pmatrix} . \end{aligned} \tag{3.95}$$

Thus it follows immediately, that $m(w)$ or equivalently the $A_\pm(k)$ form a complete set of integrals of motion. Let us verify the symplectic structure in these variables. In terms of the original fields φ the Poisson structure (2.57) reduces to

$$\{\varphi(x)\,,\, \partial_t \varphi(y)\} \;=\; \frac{1}{x}\, \delta(x\!-\!y)\; .$$

With (3.92) this translates into

$$\{A_+(k_1)\,, A_-(k_2)\} \;=\; -\mathrm{i}\, \delta(k_1\!-\!k_2)\;, \qquad \{A_\pm(k_1)\,, A_\pm(k_2)\} \;=\; 0\; . \tag{3.96}$$

For the Fourier transforms appearing in (3.95) this implies

$$\left\{ \int_0^\infty dk_1\, A_+(k_1) e^{+\mathrm{i}k_1 v}\;,\; \int_0^\infty dk_2\, A_-(k_2) e^{-\mathrm{i}k_2 w} \right\} \;=\; \frac{1}{v-w}\;,$$

and

$$\{m(v)\,, m(w)\} \;=\; \frac{2}{v-w}\; . \tag{3.97}$$

Upon exponentiation, this leads to the abelian version of (3.60), (3.61) and (3.62).

Moreover the action of the Geroch group takes a simple form in this abelian case. According to (3.90), in the abelian case we expect a symplectic action which is generated by

$$G[\Lambda] \;=\; \oint_\ell \frac{dv}{2\pi\mathrm{i}}\, \Lambda(v) \mathrm{ad}_{m(v)}\;,$$

with some function $\Lambda(v)$. In the representation (3.93) this is easily seen to give rise to

$$G[\Lambda]\,\varphi \;=\; \oint_\ell \frac{dw}{2\pi\mathrm{i}}\; \Lambda(w)\, \gamma^{-1} \partial_w \gamma(w)\; . \tag{3.98}$$

This coincides with the abelian version of (3.74).

Quantization of the abelian sector is straightforward [3]. The Poisson algebra (3.96) gives rise to a representation in terms of creation and annihilation operators

$$A_- |0\rangle = 0 \qquad \text{with} \qquad A_+ = A_-^\dagger \, . \tag{3.99}$$

Coherent quantum states may be constructed basically in the same way as in flat space quantum field theory. However, a recent discussion has shown that, interestingly enough, these states do not provide coherence of all essential physical quantities [4, 46]. Even though we know that the linearized structure (3.96) does not appear in the full nonabelian model, (3.99) may give a hint on the nature of relevant representations of the operator algebra which replaces the integrals of motion after quantization. We will return to this point in section 5.3.

4 Supergravity

In this chapter, we show how the results obtained so far may be extended to locally supersymmetric theories [105]. The simplest of these models descends from dimensional reduction of $N=1$ supergravity in four dimensions and leads to an $N=2$ superextension of the bosonic model described in section 2.1 (see e.g. [99]).

Here, we analyze maximally extended $N=16$ supergravity in two dimensions. This is the theory obtained by Kaluza-Klein type dimensional reduction from $N=1$ supergravity in eleven dimensions [21] via $N=8$ supergravity in four dimensions [22] and via the $N=16$ theory in three dimensions [92]. A detailed description of the dimensional reduction to two dimensions has been given in [62, 98, 103].

After introducing the model, we extend the canonical framework of section 2.3 to the fermionic sector. We give the expressions for the generators of local supersymmetries in all fermionic orders and work out the full $N=16$ superconformal constraint algebra which extends the conformal algebra (2.62) of the bosonic sector. Finally, we construct nonlocal charges associated to the linear system. Generalizing (3.26), they are shown to be invariant under local supersymmetry and hence under the full constraint superalgebra. The Poisson algebra of charges turns out to coincide with the structures that already appeared in the bosonic sector.

4.1 The model: $N=16$ supergravity in two dimensions

In this section, we describe the superextension of the bosonic model that we have treated in the previous chapters and set up the canonical framework.

Let us state the field content of $d=2, N=16$ supergravity. The matter sector consists of 128 bosons and 128 fermions which transform in inequivalent (left and right handed) spinor representations of $SO(16)$. The bosonic fields form the coset space $\mathbf{G}/\mathbf{H} = E_{8(+8)}/SO(16)$, i.e. they are encoded in a matrix $\mathcal{V} \in E_8$ with $SO(16)$ gauge freedom (2.38). We denote the generators of the Lie algebra $\mathfrak{e}_8$ by $X^{IJ} = -X^{JI}$ with $I, J = 1, \ldots, 16$ and Y^A with $A = 1, \ldots, 128$, corresponding to the decomposition $\mathbf{248} \to \mathbf{120} \oplus \mathbf{128}$ of $\mathfrak{e}_8$ into the adjoint and the fundamental spinor representation of $SO(16)$. The defining relations of $\mathfrak{e}_8$ are

$$\begin{aligned} \left[X^{IJ}, X^{KL}\right] &= \delta^{JK} X^{IL} - \delta^{IK} X^{JL} + \delta^{IL} X^{JK} - \delta^{JL} X^{IK} , \\ \left[X^{IJ}, Y^A\right] &= -\tfrac{1}{2} \Gamma^{IJ}_{AB} Y^B , \qquad \left[Y^A, Y^B\right] = \tfrac{1}{4} \Gamma^{IJ}_{AB} X^{IJ} , \end{aligned} \tag{4.1}$$

where the Γ^{IJ}_{AB} denote the $SO(16)$-Γ-matrices

$$\Gamma^I_{A\dot{A}} \Gamma^J_{\dot{A}B} = \delta^{IJ}_{AB} + \Gamma^{IJ}_{AB} . \tag{4.2}$$

In the adjoint representation of $\mathfrak{e}_8$ these generators are normalized such that

$$\mathrm{tr}\left(X^{IJ}X^{KL}\right) = -120\,\delta^{IJ}_{KL}\;, \qquad \mathrm{tr}\left(Y^AY^B\right) = 60\,\delta^{AB}\;.$$

The full coset structure of the bosonic sector has been described in section 2.2. According to (2.39) the bosonic current $\mathcal{V}^{-1}\partial_\mu\mathcal{V}$ is decomposed into

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} = \tfrac{1}{2}Q^{IJ}_\mu X^{IJ} + P^A_\mu Y^A\;, \tag{4.3}$$

exhibiting the $SO(16)$ gauge field Q^{IJ}_μ and the P^A_μ transforming in the left handed spinor representation of $SO(16)$. The fermionic matter part is given by 128 physical fermions which accordingly transform in the right handed spinor representation of $SO(16)$; they are denoted by $\chi^{\dot{A}}_\pm$ with $\dot{A} = 1,\ldots,128$.

In addition, we have the gravitino ψ^I_α and the "dilatino" ψ^I_2 which descend from the $3d$ gravitino and form the superpartners of the zweibein e^α_μ and the dilaton ρ, respectively (cf. (2.5),(2.6)). Before we state the Lagrangian, we introduce our spinor conventions in two dimensions.

Spinor conventions We introduce γ-matrices in two-dimensions which satisfy the algebra (in flat indices α, β)

$$\gamma_\alpha\gamma_\beta = \eta_{\alpha\beta} + \epsilon_{\alpha\beta}\gamma^3 \quad , \quad \gamma^3\gamma_\alpha = \epsilon_{\alpha\beta}\gamma^\beta \tag{4.4}$$

with $\epsilon_{01} = -\epsilon^{01} = 1$. An explicit realization is given by

$$\gamma_0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\;, \quad \gamma_1 = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}\;, \quad \gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\;. \tag{4.5}$$

We make use of the Majorana representation where the charge conjugation matrix is $\mathcal{C} = \gamma_0$, such that a Majorana spinor obeying $\overline{\psi} = \psi^T\mathcal{C}$ has two real components. We will use the decomposition into Majorana-Weyl spinors

$$\tfrac{1}{2}(1 \pm \gamma^3)\psi \;\equiv\; \begin{pmatrix} \psi_\pm \\ \pm\psi_\pm \end{pmatrix}\;, \tag{4.6}$$

and treat the one component spinors $\psi_\pm$ as real anticommuting variables at the classical level. Let us also give some useful rules for the transcription between two component and one component notation:

$$\begin{aligned} \overline{\psi}\chi &= 2\mathrm{i}(\psi_+\chi_- - \psi_-\chi_+) \qquad \overline{\psi}\gamma^3\chi = -2\mathrm{i}(\psi_+\chi_- + \psi_-\chi_+) \\ \overline{\psi}\gamma_+\chi &= 2\psi_+\chi_+ \qquad \overline{\psi}\gamma_-\chi = 2\psi_-\chi_- \end{aligned}$$

The fully covariant derivatives on the spinor fields are given by

$$\begin{aligned} D_\mu\psi^I &= \partial_\mu\psi^I + \tfrac{1}{4}\omega_{\mu\alpha\beta}\epsilon^{\alpha\beta}\gamma^3\psi^I + Q^{IJ}_\mu\psi^J\;, \\ D_\mu\chi^{\dot{A}} &= \partial_\mu\chi^{\dot{A}} + \tfrac{1}{4}\omega_{\mu\alpha\beta}\epsilon^{\alpha\beta}\gamma^3\chi^{\dot{A}} + \tfrac{1}{4}Q^{IJ}_\mu\Gamma^{IJ}_{\dot{A}\dot{B}}\chi^{\dot{B}}\;, \end{aligned} \tag{4.7}$$

where the spin connection $\omega_{\mu\alpha\beta}$ is a function of the two-dimensional metric and its superpartners [106].

Lagrangian and equations of motion

The Lagrangian of $d=2, N=16$ supergravity is most conveniently obtained by dimensional reduction of $d=3, N=16$ supergravity [92] as described in [98].

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{2}\rho E^{(2)}R^{(2)} + 2\rho E^{(2)}\epsilon^{\mu\nu}\overline{\psi}_2^I D_\mu\psi_\nu^I - \mathrm{i}\rho E^{(2)}\overline{\chi}^{\dot{A}}\gamma^\mu D_\mu\chi^{\dot{A}} + \tfrac{1}{2}\rho E^{(2)}P^{\mu A}P_\mu^A \\ &\quad - \rho E^{(2)}\overline{\chi}^{\dot{A}}\gamma^\nu\gamma^\mu\psi_\nu^I\Gamma^I_{A\dot{A}}P_\mu^A - \mathrm{i}\rho E^{(2)}\overline{\chi}^{\dot{A}}\gamma^3\gamma^\mu\psi_2^I\Gamma^I_{A\dot{A}}P_\mu^A\ ,\end{aligned} \tag{4.8}$$

up to higher order fermionic terms. The first two terms of (4.8) describe two-dimensional gravity and the $N=16$ Rarita-Schwinger extension. The next two terms give the matter couplings of the 128 fermionic and 128 bosonic fields, respectively; the last terms are of Noether type to ensure supersymmetry of the action.

In addition, there arise several quartic fermionic terms which we omit here. Although in principle they may be determined from the higher-dimensional theory, this computation becomes rather lengthy due to additional fermionic contributions which arise from the elimination of the Kaluza-Klein vector fields.[10] Nonetheless, in (4.24) below we give the exact expressions for the generators of the local supersymmetries, which are sufficient to reconstruct all higher order terms systematically as well as to prove exact supersymmetry of the conserved charges.

The action (4.8) is manifestly invariant under general coordinate transformation in two dimensions, as well as under the $SO(16)$ transformations

$$\begin{aligned}\delta_\omega Q_\pm^{IJ} &= D_\pm\omega^{IJ} = \partial_\mu\omega^{IJ} + Q_\mu^{IK}\omega^{KJ} - Q_\mu^{JK}\omega^{KI}\ , \\ \delta_\omega P_\pm^A &= \tfrac{1}{4}\Gamma^{IJ}_{AB}\omega^{IJ}P_\pm^B\ , \\ \delta_\omega\psi^I &= \omega^{IJ}\psi^J\ , \\ \delta_\omega\chi^{\dot{A}} &= \tfrac{1}{4}\Gamma^{IJ}_{\dot{A}\dot{B}}\omega^{IJ}\chi^{\dot{B}}\ ,\end{aligned} \tag{4.9}$$

with the $SO(16)$-parameter $\omega^{IJ}(x) = -\omega^{JI}(x)$.

In the following we employ the superconformal gauge

$$e_\mu^{\ \alpha} = \delta_\mu^\alpha\,\exp\sigma\ , \qquad \psi_\mu^I = \mathrm{i}\gamma_\mu\psi^I\ , \tag{4.10}$$

which naturally extends (2.8). In this gauge, the two-dimensional spin-connection from (4.7) reads (up to bilinear fermionic terms)

$$\omega_{\pm\alpha\beta} = \mp\epsilon_{\alpha\beta}\partial_\pm\sigma\ ,$$

such that in terms of the the one-component spinors introduced above, the covariant derivative

$$\left(\partial_\pm \pm \tfrac{1}{4}\omega_{\pm\alpha\beta}\epsilon^{\alpha\beta}\right)\psi_\mp = \left(\partial_\pm + \tfrac{1}{2}\partial_\pm\sigma\right)\psi_\mp = \partial_\pm\left(\psi_\mp\exp(\tfrac{1}{2}\sigma)\right)\ ,$$

may be absorbed by rescaling the fermions with the conformal factor. Like in the bosonic case, the conformal factor then almost completely disappears from the Lagrangian except for its explicit appearance in the two-dimensional curvature term coupled to the dilaton ρ.

[10] Unlike in (2.16), here, the Kaluza-Klein field strengths do not vanish but are expressed through bilinear fermionic terms. Their elimination from the Lagrangian then gives rise to additional quartic fermionic terms.

We next list the equations of motion in the superconformal gauge. The bosonic equations (2.48), (2.49) are extended to

$$\begin{aligned} D_+(\rho P_-^A) + D_-(\rho P_+^A) &= 2\mathrm{i}\Gamma^I_{A\dot{A}} D_-(\rho\psi^I_{2\,+}\chi^{\dot{A}}_+) - 2\mathrm{i}\Gamma^I_{A\dot{A}} D_+(\rho\psi^I_{2\,-}\chi^{\dot{A}}_-) \\ &\quad + 2\mathrm{i}\rho\Gamma^{IJ}_{AB} P^B_- \psi^I_{2\,+}\psi^J_+ - 2\mathrm{i}\rho\Gamma^{IJ}_{AB} P^B_+ \psi^I_{2\,-}\psi^J_- \\ &\quad + \tfrac{\mathrm{i}}{4}\rho\Gamma^{IJ}_{AB}\Gamma^{IJ}_{\dot{A}\dot{B}} P^B_- \chi^{\dot{A}}_+\chi^{\dot{B}}_+ + \tfrac{\mathrm{i}}{4}\rho\Gamma^{IJ}_{AB}\Gamma^{IJ}_{\dot{A}\dot{B}} P^B_+ \chi^{\dot{A}}_-\chi^{\dot{B}}_- \ , \\ \partial_+\partial_-\widehat{\sigma} \ = \ \partial_+\partial_-\sigma &= -\tfrac{1}{2}P^A_+P^A_- - \mathrm{i}\left(\chi^{\dot{A}}_+ D_-\chi^{\dot{A}}_+ + \chi^{\dot{A}}_- D_+\chi^{\dot{A}}_-\right) \ , \end{aligned} \tag{4.11}$$

modulo quartic spinor terms. The fermionic equations of motion read

$$\begin{aligned} D_\pm(\rho^{\frac{1}{2}}\chi^{\dot{A}}_\mp) &= \mp\tfrac{1}{2}\rho^{\frac{1}{2}}\psi^I_{2\,\mp}\Gamma^I_{A\dot{A}}P^A_\pm \ , \\ D_\pm\psi^I_\mp &= -\tfrac{1}{2}\chi^{\dot{A}}_\mp\Gamma^I_{A\dot{A}}P^A_\pm \ , \\ D_\pm(\rho\psi^I_{2\,\mp}) &= 0 \ , \end{aligned} \tag{4.12}$$

modulo cubic spinor terms.

Like in the bosonic case, there are further equations that descend from the Lagrangian (4.8) before (super)conformal gauge is adopted. They are to be regarded as constraints arising with the unimodular components of the $2d$ metric and the traceless modes of the gravitino, respectively, as Lagrangian multipliers. The resulting expressions are

$$\begin{aligned} T_{\pm\pm} &= -\rho P^A_\pm P^A_\pm + 2\partial_\pm\rho\,\partial_\pm\hat{\sigma} \pm 2\mathrm{i}\rho P^A_\pm\Gamma^I_{A\dot{B}}\psi^I_{2\,\pm}\chi^{\dot{B}}_\pm + 2\mathrm{i}\rho\chi^{\dot{A}}_\pm D_\pm\chi^{\dot{A}}_\pm \\ &\quad \pm 2\mathrm{i}\psi^I_\pm D_\pm(\rho\psi^I_{2\,\pm}) \pm 2\mathrm{i}\rho\psi^I_{2\,\pm} D_\pm(\psi^I_\pm) \qquad\qquad \approx \ 0 \ , \end{aligned} \tag{4.13}$$

$$S^I_\pm \ = \ -2D_\pm(\rho\psi^I_{2\,\pm}) + 2\rho\partial_\pm\sigma\,\psi^I_{2\,\pm} \pm 2\rho\chi^{\dot{A}}_\pm\Gamma^I_{A\dot{A}}P^A_\pm \mp 2\partial_\pm\rho\,\psi^I_\pm \quad \approx \ 0 \ , \tag{4.14}$$

generating conformal and superconformal transformations. Modulo higher order fermionic terms the superconformal transformations of the fields are given by

$$\begin{aligned} \mathcal{V}^{-1}\delta_\pm\mathcal{V} &= \mp 2\mathrm{i}\epsilon^I_\pm\chi^{\dot{A}}_\pm\Gamma^I_{\dot{A}A}Y^A \ , & \delta_\pm\chi^{\dot{A}}_\pm &= \mp\epsilon^I_\pm\Gamma^I_{A\dot{A}}P^A_\pm \ , \\ \delta_\pm\rho &= 2\mathrm{i}\rho\epsilon^I_\pm\psi^I_{2\,\pm} \ , & \delta_\pm\psi^I_{2\,\pm} &= \rho^{-1}\partial_\pm\rho\epsilon^I_\pm \ , \\ \delta_\pm\sigma &= \mp 2\mathrm{i}\epsilon^I_\pm\psi^I_\pm \ , & \delta_\pm\psi^I_\pm &= \mp\left(D_\pm\epsilon^I_\pm + \partial_\pm\sigma\epsilon^I_\pm\right) \ , \end{aligned} \tag{4.15}$$

with the parameter $\epsilon^I_\pm$ obeying

$$D_\pm\epsilon^I_\mp = 0 \tag{4.16}$$

again modulo cubic spinor terms. These are the supersymmetry transformations which leave the Lagrangian (4.8) invariant and are moreover compatible with the superconformal gauge choice (4.10). As an algebra, these transformations close into an $N = 16$ superconformal algebra which additionally contains the conformal and the local $SO(16)$ transformations. It is distinguished from the standard superconformal algebras by the fact that it is a soft algebra, i.e. it appears with field dependent structure “constants”. This will be discussed in more detail in the next section.

Canonical Poisson brackets

The Poisson brackets of the bosonic sector of the model are obtained in the same way as (2.50) and (2.57) above. With canonical momenta

$$\Pi^{IJ} \equiv \frac{\delta S}{\delta(\partial_0 Q_1^{IJ})} \,, \qquad \Pi^{A} \equiv \frac{\delta S}{\delta(\partial_0 P_1^{A})} \,,$$

the relations (2.54) and (2.55) receive additional fermionic contributions and in components take the following form

$$P_0^A \;=\; -\frac{1}{\rho}\left(D_1\Pi^A + \tfrac{1}{4}\Pi^{IJ}\Gamma^{IJ}_{AB}P_1^B\right) - \mathrm{i}\Gamma^I_{A\dot{A}}\psi^I_{2+}\chi^{\dot{A}}_+ Y^A + \mathrm{i}\Gamma^I_{A\dot{A}}\psi^I_{2-}\chi^{\dot{A}}_- Y^A \,, \tag{4.17}$$

and

$$\begin{aligned}\Phi^{IJ} \;&=\; D_1\Pi^{IJ} + \tfrac{1}{4}\Gamma^{IJ}_{AB}P_1^A\Pi^B \\ &\quad - 2\mathrm{i}\rho\left(\psi_+^{[I}\psi^{J]}_{2+} - \psi_-^{[I}\psi^{J]}_{2-}\right) - \tfrac{\mathrm{i}}{4}\rho\Gamma^{IJ}_{\dot{A}\dot{B}}\left(\chi^{\dot{A}}_+\chi^{\dot{B}}_+ + \chi^{\dot{A}}_-\chi^{\dot{B}}_-\right) \;\approx\; 0 \,.\end{aligned} \tag{4.18}$$

The first relation gives rise to the bosonic Poisson brackets for the physical fields, the second one defines the set of weakly vanishing first-class constraints generating the $SO(16)$ transformations. In analogy to (2.57) we obtain the Poisson brackets

$$\begin{aligned}
\left\{P^A_\pm(x)\,,\mathcal{V}(y)\right\} \;&=\; -\frac{1}{2\rho}\mathcal{V}(x)Y^A\,\delta(x-y)\,, \\
\left\{P^A_\pm(x)\,,Q_1^{IJ}(y)\right\} \;&=\; \frac{1}{8\rho}\,\Gamma^{IJ}_{AB}P_1^B\,\delta(x-y)\,, \\
\left\{P^A_\pm(x)\,,P^B_\pm(y)\right\} \;&=\; \pm\frac{1}{8\rho}\,\Gamma^{IJ}_{AB}Q_1^{IJ}\,\delta(x-y) \pm \frac{1}{4}\left(\frac{1}{\rho(x)}+\frac{1}{\rho(y)}\right)\,\delta^{AB}\,\delta'(x-y) \\
&\quad + \frac{\mathrm{i}}{16\rho}\,\Gamma^{IJ}_{AB}\Gamma^{IJ}_{\dot{A}\dot{B}}\left(\chi^{\dot{A}}_+\chi^{\dot{B}}_+ + \chi^{\dot{A}}_-\chi^{\dot{B}}_-\right)\,\delta(x-y) \\
&\quad + \frac{\mathrm{i}}{2\rho}\Gamma^{IJ}_{AB}\left(\psi^I_+\psi^J_{2+} - \psi^I_-\psi^J_{2-}\right)\delta(x-y) \\
&\quad + \frac{\mathrm{i}}{4\rho}\Gamma^{IJ}_{AB}\left(\psi^I_{2+}\psi^J_{2+} + \psi^I_{2-}\psi^J_{2-}\right)\delta(x-y) \\
&\quad - \frac{1}{8\rho^2}\,\Gamma^{IJ}_{AB}\Phi_{IJ}\,\delta(x-y)\,, \\
\left\{P^A_\pm(x)\,,P^B_\mp(y)\right\} \;&=\; \mp\frac{1}{4\rho^2}\,\partial_1\rho\,\delta^{AB}\,\delta(x-y) - \frac{1}{8\rho^2}\,\Gamma^{IJ}_{AB}\Phi_{IJ}\,\delta(x-y) \\
&\quad + \frac{\mathrm{i}}{16\rho}\,\Gamma^{IJ}_{AB}\Gamma^{IJ}_{\dot{A}\dot{B}}\left(\chi^{\dot{A}}_+\chi^{\dot{B}}_+ + \chi^{\dot{A}}_-\chi^{\dot{B}}_-\right)\,\delta(x-y) \\
&\quad + \frac{\mathrm{i}}{2\rho}\Gamma^{IJ}_{AB}\left(\psi^I_+\psi^J_{2+} - \psi^I_-\psi^J_{2-}\right)\delta(x-y) \\
&\quad + \frac{\mathrm{i}}{4\rho}\Gamma^{IJ}_{AB}\left(\psi^I_{2+}\psi^J_{2+} + \psi^I_{2-}\psi^J_{2-}\right)\delta(x-y)\,,
\end{aligned} \tag{4.19}$$

$$\begin{aligned}
\left\{P^A_\pm(x)\,,\partial_\pm\sigma(y)\right\} \;&=\; \frac{1}{4\rho}\left(P_0^A + \mathrm{i}\Gamma^I_{A\dot{B}}\left(\psi^I_{2+}\chi^{\dot{B}}_+ - \psi^I_{2-}\chi^{\dot{B}}_-\right)\right)\delta(x-y)\,, \\
\left\{P^A_\pm(x)\,,\partial_\mp\sigma(y)\right\} \;&=\; \frac{1}{4\rho}\left(P_0^A + \mathrm{i}\Gamma^I_{A\dot{B}}\left(\psi^I_{2+}\chi^{\dot{B}}_+ - \psi^I_{2-}\chi^{\dot{B}}_-\right)\right)\delta(x-y)\,.
\end{aligned} \tag{4.20}$$

The fermionic sector as usual requires a Dirac procedure since the fermionic canonical momenta appear proportional to the fermions themselves. The final brackets are found to be

$$\begin{aligned}
\left\{\chi^{\dot A}_\pm(x)\,,\chi^{\dot B}_\pm(y)\right\} &= \frac{\mathrm{i}}{4\rho}\,\delta^{\dot A\dot B}\,\delta(x-y)\;, \\
\left\{\psi^I_\pm(x)\,,\psi^J_{2\,\pm}(y)\right\} &= \pm\frac{\mathrm{i}}{4\rho}\,\delta^{IJ}\;\delta(x-y)\;.
\end{aligned} \tag{4.21}$$

Due to the explicit appearance of the dilaton field in the r.h.s. of (4.21), this fermionic Dirac procedure also gives rise to the following non-vanishing mixed brackets

$$\begin{aligned}
\left\{\partial_0\sigma(x)\,,\chi^{\dot A}_\pm(y)\right\} &= -\frac{1}{2\rho}\,\chi^{\dot A}_\pm\,\delta(x-y)\;, \\
\left\{\partial_0\sigma(x)\,,\psi^I_{2\,\pm}(y)\right\} &= -\frac{1}{\rho}\,\psi^I_{2\,\pm}\,\delta(x-y)\;,
\end{aligned} \tag{4.22}$$

while the form of P_0 in (4.17) gives rise to

$$\begin{aligned}
\left\{P^A_0(x)\,,\chi^{\dot B}_\pm(y)\right\} &= \mp\frac{1}{2\rho}\Gamma^I_{A\dot B}\psi^I_{2\,\pm}\,\delta(x-y)\;, \\
\left\{P^A_0(x)\,,\psi^I_\pm(y)\right\} &= \frac{1}{2\rho}\Gamma^I_{A\dot B}\chi^{\dot B}_\pm\,\delta(x-y)\;.
\end{aligned}$$

Since most of these brackets look rather unwieldy, it may be worthwhile to look for simpler canonical variables. E.g. the modified momenta

$$\widetilde{P}^A_0 \;:=\; \rho\,P^A_0 - 2\mathrm{i}\rho\,\psi^I_{2+}\gamma_1\chi^{\dot A}_+\Gamma^I_{A\dot A} + 2\mathrm{i}\rho\,\psi^I_{2-}\gamma_1\chi^{\dot A}_-\Gamma^I_{A\dot A}\;, \tag{4.23}$$

commute with all the fermions and with $\partial_\pm\sigma$. Moreover, we notice that the rescaled fermions $\rho\psi$ and $\rho^{\frac{1}{2}}\chi$ commute with $\partial_0\sigma$ as well.

4.2 Constraint superalgebra

In this section, we establish the constraint superalgebra generated by the superconformal transformations (4.15). As discussed above, this is the part of the original symmetry algebra of (4.8) which is compatible with the truncation to superconformal gauge (4.10). These transformations close into an $N=16$ superconformal algebra which in addition contains the conformal transformations generated by (4.13) and the $SO(16)$ gauge transformations (4.9). Closure of the supersymmetry algebra is known from general reasoning [106, 92].

To avoid overlap with the general discussion of the constraint algebra in the bosonic case, we simply state the full expressions for the supersymmetry generators

$$\begin{aligned}
S^I_\pm \;=\;& \pm D_1(\rho\psi^I_{2\,\pm}) - \rho\partial_\pm\sigma\,\psi^I_{2\,\pm} \mp \rho\chi^{\dot A}_\pm\Gamma^I_{A\dot A}P^A_\pm \pm \partial_\pm\rho\,\psi^I_\pm \\
& \mp \mathrm{i}\rho\psi^J_\pm\,\chi_\pm\Gamma^{IJ}\chi_\pm - \tfrac{\mathrm{i}}{2}\,\rho\psi^J_{2\,\pm}\Gamma^{IJ}_{\dot A\dot B}\left(\chi^{\dot A}_\pm\chi^{\dot B}_\pm - \chi^{\dot A}_\mp\chi^{\dot B}_\mp\right) \\
& + 2\mathrm{i}\rho\psi^I_\pm\psi^J_\pm\psi^J_{2\,\pm} \pm 2\mathrm{i}\rho\psi^I_\mp\psi^J_{2\,\pm}\psi^J_{2\,\mp} \mp 2\mathrm{i}\rho\psi^I_{2\,\mp}\psi^J_\mp\psi^J_{2\,\pm} - 2\mathrm{i}\rho\psi^I_{2\,\mp}\psi^J_{2\,\pm}\psi^J_{2\,\mp}\;,
\end{aligned} \tag{4.24}$$

including all cubic fermionic terms. These terms have been reconstructed from the requirement of closure of their algebra

$$\begin{aligned}\left\{S^I_\pm(x)\,,S^J_\pm(y)\right\} &= -\delta^{IJ}\Big(\mathrm{i}T_{\pm\pm}\mp 2\psi^K_\pm S^K_\pm - \tfrac14\chi^{\dot A}_\pm\chi^{\dot B}_\pm\Gamma^{KL}_{\dot A\dot B}\Phi_{KL}\Big)\,\delta(x-y) \\ &\quad \mp\Big(\psi^I_\pm S^J_\pm + \psi^J_\pm S^I_\pm\Big)\,\delta(x-y) \\ &\quad + \tfrac12\chi^{\dot A}_\pm\chi^{\dot B}_\pm\Big(\Gamma^{IK}_{\dot A\dot B}\Phi^{KJ}+\Gamma^{JK}_{\dot A\dot B}\Phi^{KI}\Big)\,\delta(x-y)\;,\end{aligned} \tag{4.25}$$

$$\begin{aligned}\left\{S^I_+(x)\,,S^J_-(y)\right\} &= -\delta^{IJ}\Big(\psi^K_{2+}S^K_- + \psi^K_{2-}S^K_+\Big)\,\delta(x-y) \\ &\quad + \Big(\psi^I_{2-}S^J_+ + \psi^J_{2+}S^I_-\Big)\,\delta(x-y) \\ &\quad + \tfrac14\chi^{\dot A}_+\chi^{\dot B}_-\,\Gamma^I_{\dot AA}\Gamma^{KL}_{AB}\Gamma^J_{B\dot B}\,\Phi^{KL}\,\delta(x-y)\;.\end{aligned} \tag{4.26}$$

This again is an exact result, i.e. valid in all fermionic orders. The constraint superalgebra consistently closes in terms of the Virasoro constraints $T_{\pm\pm}$ and the $SO(16)$ constraints Φ^{IJ}. We emphasize, that the closure of this algebra uniquely fixes all the cubic fermionic terms in (4.24).

The supersymmetry generators (4.24) are the crucial operators here, since they span the full constraint algebra. Thus, complete knowledge of these generators is sufficient to prove gauge invariance of the nonlocal conserved charges in the next section. Moreover, with (4.24) at hand we are in position to compute e.g. the quartic spinorial contributions to $T_{\pm\pm}$ straight-forwardly. By means of the super-Jacobi identities

$$\{\{S^I_\pm,S^J_\pm\},\varphi\} \;=\; \{S^I_\pm,\{S^J_\pm,\varphi\}\}+\{S^J_\pm,\{S^I_\pm,\varphi\}\}\;,$$

we can further directly obtain the conformal transformations generated by the $T_{\pm\pm}$ in all fermionic orders.

With this in mind, we restrict to giving the rest of the superconformal algebra only up to higher order fermionic terms again:

$$\begin{aligned}\{T_{\pm\pm}(x)\,,T_{\pm\pm}(y)\} &= \mp\left(T_{\pm\pm}(x)+T_{\pm\pm}(y)\right)\,\delta'(x-y)\;, \\ \{T_{\pm\pm}(x)\,,T_{\mp\mp}(y)\} &= \tfrac14\,\Gamma^{IJ}_{AB}P^A_+P^B_-\Phi^{IJ}\,\delta(x-y)\;, \\ \left\{T_{\pm\pm}(x)\,,S^I_\pm(y)\right\} &= \mp\tfrac32 S^I_\pm(y)\,\delta'(x-y)+D_\pm S^I_\pm\,\delta(x-y) \\ &\quad \mp\tfrac14\,\Gamma^{KL}_{AB}\Gamma^I_{B\dot A}P^A_\pm\chi^{\dot A}_\pm\Phi^{KL}\,\delta(x-y)\;, \\ \left\{T_{\pm\pm}(x)\,,S^I_\mp(y)\right\} &= \pm\tfrac14\,\Gamma^{KL}_{AB}\Gamma^I_{B\dot A}P^A_\pm\chi^{\dot A}_\mp\Phi^{KL}\,\delta(x-y)\;, \\ \left\{\Phi^{IJ}(x)\,,S^K_\pm(y)\right\} &= \tfrac12\left(\delta^{IK}S^J(x)-\delta^{JK}S^I(x)\right)\,\delta(x-y)\;, \\ \left\{\Phi^{IJ}(x)\,,T_{\pm\pm}(y)\right\} &= 0\;, \\ \left\{\Phi^{IJ}(x)\,,\Phi^{KL}(y)\right\} &= \Big(\delta^{JK}\Phi^{IL}-\delta^{IK}\Phi^{JL}+\delta^{IL}\Phi^{JK}-\delta^{JL}\Phi^{IK}\Big)\,\delta(x-y)\;.\end{aligned} \tag{4.27}$$

The gauge transformations (4.15) and (4.9) are generated by

$$\delta_\pm\varphi = 2\mathrm{i}\int dx\,\epsilon^I_\pm(x)\left\{S^I_\pm(x)\,,\,\varphi\right\}\;,\quad\text{and}\quad \delta_\omega\,\varphi \equiv \int dx\;\tfrac12\omega^{IJ}(x)\left\{\Phi^{IJ}(x)\,,\varphi\right\}\;,$$

respectively. Conformal coordinate transformations with parameter $\xi^{\pm} = \xi^{\pm}(x^{\pm})$ are generated as in the bosonic model (2.60). One can verify that again $T_{\pm\pm}$ generates translations along the $x^{\pm}$ coordinates modulo a local $SO(16)$ transformation with field dependent parameter Q_1^{IJ}.

Remark 4.1 For computation of the canonical Poisson brackets it is necessary to rewrite $T_{\pm\pm}$ entirely expressed in terms of canonical variables. I.e. the time derivatives of the fermions in (4.13) must be expressed by their spatial derivatives, making use of the fermionic equations of motion (4.12)

$$\begin{aligned} D_{\pm}(\rho^{\frac{1}{2}}\chi_{\pm}^{\dot{A}}) &= \pm D_1(\rho^{\frac{1}{2}}\chi_{\pm}^{\dot{A}}) \pm \tfrac{1}{2}\rho^{\frac{1}{2}}\psi_{2\,\pm}^{I}\Gamma^{I}_{A\dot{A}}P_{\mp}^{A} \;, \\ D_{\pm}\psi_{\pm}^{I} &= \pm D_1\psi_{\pm}^{I} - \tfrac{1}{2}\chi_{\pm}^{\dot{A}}\Gamma^{I}_{A\dot{A}}P_{\mp}^{A} \;, \\ D_{\pm}(\rho\psi_{2\,\pm}^{I}) &= \pm D_1(\rho\psi_{2\,\pm}^{I}) \;, \end{aligned}$$

where the l.h.s. exhibits conformal covariance whereas the r.h.s consists of canonical variables. The “canonical” form (in contrast to the covariant form (4.13)) of the energy-momentum constraint is then given by

$$\begin{aligned} T_{\pm\pm} &= -\rho P_{\pm}^{A}P_{\pm}^{A} + 2\partial_{\pm}\rho\,\partial_{\pm}\sigma \mp \partial_1\partial_{\pm}\rho \pm 2\mathrm{i}\rho P_1^{A}\Gamma^{I}_{A\dot{B}}\psi_{2\,\pm}^{I}\chi_{\pm}^{\dot{B}} \\ &\quad \pm 2\mathrm{i}\rho\chi_{\pm}^{\dot{A}}D_1\chi_{\pm}^{\dot{A}} + 2\mathrm{i}\psi_{\pm}^{I}D_1(\rho\psi_{2\,\pm}^{I}) + 2\mathrm{i}\rho\psi_{2\,\pm}^{I}D_1(\psi_{\pm}^{I}) \;, \end{aligned} \tag{4.28}$$

again up to quartic fermionic terms.

The constraint superalgebra (4.25), (4.26), (4.27) is a superconformal extension of the Virasoro algebra (2.62) with $N=16$ supercharges. In contrast to the superconformal algebras which have been studied in string theory and conformal field theory, it exhibits some rather unusual features. Thus, its existence does not contradict the well-known absence of superconformal algebras with $N>4$ [108].

First of all, this model does not allow the complete splitting into chiral halves: S_+ and S_- do not commute in (4.26). Another important property of (4.25) and (4.26) is, that they obviously do not close into a linear algebra in the usual sense. Rather, on the r.h.s the constraints $S_{\pm}^{I}$ appear with coefficients that explicitly involve the fermionic fields ψ^{I} and ψ_2^{I}. This is an example of the “soft” gauge algebras arising in (super)gravity [106, 115].

In addition, no internal chiral currents appear here. A linear superconformal algebra with N supercharges requires an internal bosonic $SO(N$ current. This is immediately seen from the super-Jacobi identities involving $\{S^I,\{S^J,S^K\}\}$. Vanishing of the δ' contributions necessitates the additional current. In (4.25) in contrast, these terms originate from the additional contributions due to the field dependent structure constants on the r.h.s.. The $SO(16)$-current Φ^{IJ} which is part of the gauge algebra in this model is obviously not chiral. Its fermionic part splits into contributions with conformal weights $h^+=1$ and $h^-=1$, respectively. Nonetheless, according to (4.27) the total conformal weight of Φ^{IJ} is zero. An underlying reason for this compensation is the fact, that in our model in contrast to the superconformal string theories not only the fermionic but also the bosonic fields carry $SO(16)$ charge.

We close this section by stating the super extension of the gauge fixing (2.20) of the constraint superalgebra

$$\rho^{+} = x^{+} \;, \quad \rho^{-} = \pm x^{-} \;, \qquad \psi_2^{I} = 0 \;, \tag{4.29}$$

which may accordingly be referred to as the super-Weyl gauge. Indeed this completely fixes the conformal and superconformal gauge freedom.

4.3 Nonlocal charges and their Poisson algebra

In this section we show that supersymmetric nonlocal conserved charges may be constructed in the same way as for the bosonic case studied in the previous chapters. The starting point is the extension of the linear system (3.1) given in [98, 103]. With the full generators of $N=16$ supersymmetry (4.24) at hand we show that this linear system does not receive any quartic fermionic corrections but already generates the equations of motion into all orders. The charges extracted from the transition matrices are invariant under the full gauge algebra (4.27). Finally, we find that the Poisson algebra of charges coincides with the one obtained in the bosonic sector (3.60), (3.61), (3.62).

Linear system

The supergravity equations of motion can be obtained as the compatibility condition of the following extension [98, 103] of the linear system (3.1) for an E_8-valued matrix $\widehat{\mathcal{V}}$:

$$\widehat{\mathcal{V}}^{-1}(\gamma)\,\partial_\pm\widehat{\mathcal{V}}(\gamma) \;=\; L_\pm(\gamma) \;\equiv\; \tfrac{1}{2}\widehat{Q}^{IJ}_\pm(\gamma)X^{IJ} + \widehat{P}^A_\pm(\gamma)Y^A \;, \tag{4.30}$$

with the connection

$$\begin{aligned}
\widehat{Q}^{IJ}_\pm(\gamma) &= Q^{IJ}_\pm - \frac{2\mathrm{i}\gamma}{(1\pm\gamma)^2}\,\Big(8\psi^{[I}_{2\,\pm}\psi^{J]}_\pm \pm \Gamma^{IJ}_{\dot A\dot B}\chi^{\dot A}_\pm\chi^{\dot B}_\pm\Big) - \frac{32\mathrm{i}\gamma^2}{(1\pm\gamma)^4}\,\psi^I_{2\,\pm}\psi^J_{2\,\pm} \;, \\
\widehat{P}^A_\pm(\gamma) &= \frac{1\mp\gamma}{1\pm\gamma}\,P^A_\pm + \frac{4\mathrm{i}\gamma(1\mp\gamma)}{(1\pm\gamma)^3}\,\Gamma^I_{A\dot B}\psi^I_{2\,\pm}\chi^{\dot B}_\pm \;,
\end{aligned}$$

and the variable spectral parameter γ from (3.3).

We emphasize that despite the occurrence of higher order fermionic terms in the equations of motion, the connection of the linear system (4.30) is only quadratic in the fermions. All the higher order fermionic terms are generated from it. In super-Weyl gauge (4.29) this has explicitly been shown in [98], the general proof follows from the result (4.31) below.

Nonlocal conserved charges

Here, we extend the result (3.26) of the bosonic case to the model with local supersymmetry. The modified transition matrices $\widetilde{U}(x,y,w)$ defined in (3.19) commute with the $N=16$ supersymmetry generators under the same conditions that were already analyzed for (3.26) and (3.20).

The behavior of the transition matrices (3.17) under supersymmetry transformations is the following [105]

$$\Big\{U(x,y;w), S^I_\pm(z)\Big\} \;=\; \frac{4\gamma\,\theta(x,z,y)}{\rho(1\pm\gamma)^2}\,U(x,z;w)X^{IJ}S^J_\pm(z)U(z,y;w) \tag{4.31}$$

$$
\begin{aligned}
&\pm \; \frac{\gamma\,\theta(x,z,y)}{\rho(1-\gamma^2)}\,\chi_\pm^{\dot B}\,\Gamma^I_{B\dot B}\Gamma^{JK}_{BA}\,\Phi_{JK}U(x,z;w)Y^AU(z,y;w)\\
&\pm \; \frac{1\mp\gamma}{1\pm\gamma}\,\chi_\pm^{\dot B}\,\Gamma^I_{A\dot B}\,\big(U(x,y;w)Y^A\delta(z-y)-Y^AU(x,y;w)\delta(z-x)\big)\\
&\mp \; \frac{4\gamma}{(1\pm\gamma)^2}\,\psi^J_{2\,\pm}\,\big(U(x,y;w)X^{IJ}\delta(z-y)-X^{IJ}U(x,y;w)\delta(z-x)\big)\;,
\end{aligned}
$$

with $\theta(x,z,y)$ from (3.33) above. This result is again valid in all orders of fermions, i.e. includes all the cubic fermionic terms from (4.24). For the modified transition matrices $\widetilde{U}$ it immediately implies:

$$
\begin{aligned}
\Big\{\widetilde{U}(x,y;w),S^I_\pm(z)\Big\} \;\approx\;& \frac{2\gamma}{1\pm\gamma}\,\chi_\pm^{\dot B}\,\Gamma^I_{A\dot B}\,\big(\mathcal{V}Y^A\mathcal{V}^{-1}\big)\,\widetilde{U}(x,y;w)\,\delta(z-x)\\
&-\frac{2\gamma}{1\pm\gamma}\,\chi_\pm^{\dot B}\,\Gamma^I_{A\dot B}\,\widetilde{U}(x,y;w)\,\big(\mathcal{V}Y^A\mathcal{V}^{-1}\big)\,\delta(z-y)\\
&\pm\frac{4\gamma}{(1\pm\gamma)^2}\,\psi^J_{2\,\pm}\,\big(\mathcal{V}X^{IJ}\mathcal{V}^{-1}\big)\,\widetilde{U}(x,y;w)\,\delta(z-x)\\
&\mp\frac{4\gamma}{(1\pm\gamma)^2}\,\psi^J_{2\,\pm}\,\widetilde{U}(x,y;w)\,\big(\mathcal{V}X^{IJ}\mathcal{V}^{-1}\big)\,\delta(z-y)\;.
\end{aligned}
\tag{4.32}
$$

The r.h.s. vanishes if either the physical fields vanish, or the variable spectral parameter γ does while the fields remain regular (cf. (3.22), (3.23)). In complete analogy to the integrals of motion obtained in the bosonic sector we may hence build conserved charges from the transition matrices with fermionic contributions here.

A similar transformation behavior has been observed in the supersymmetric extension of the nonlinear σ-model [20, 23, 111, 36]. There, the bosonic nonlocal charges are invariant under global supersymmetry. In our model, invariance under the local supersymmetry is an indispensable condition for meaningful observables, since supersymmetry appears as constraint.

In particular, (4.31) implies, that the connection of the linear system (4.30) does not receive any quartic corrections but captures the equations of motion in all fermionic orders: So far, this had only been shown for the $(\bar\chi\chi)^2$ terms [98], i.e. in the super-Weyl gauge (4.29) where these are the only quartic terms arising. Since by supersymmetry transformations (4.15) any solution can be fixed to obey the super-Weyl gauge, the invariance of the linear system under supersymmetry shows that indeed no quartic corrections arise in the general case.

The rest of this section is spent for a sketch of the proof of (4.31). The general formula (3.29) yields

$$
\begin{aligned}
&U(z',x,v)\,\big\{U(x,y,v)\,,S^I_\pm(z')\big\}\,U(y,z',v) \;=\\
&\qquad\int_x^y dz\,U(z',z,v)\,\big\{L_1(z,\gamma(z,v))\,,S^I_\pm(z')\big\}\,U(z,z',v)\;.
\end{aligned}
\tag{4.33}
$$

It is straightforward although lengthy to evaluate (4.33) using the form of the supersymmetry generator (4.24) and the fundamental Poisson brackets (4.19)–(4.22). Up to the higher order terms in the fermions, this result has already been given in [101]. Thus it remains to check the cubic fermionic terms.

Throughout this calculation, there appear four different sources yielding cubic fermionic terms. First they descend from the brackets involving cubic terms in the supersymmetry generators $S^I_\pm$, second from bilinear fermionic terms in the Poisson brackets (4.19) between $P_\pm$ and $P_\pm$. Third, they arise from the Poisson brackets involving $\partial_\pm\sigma$ in $S^I_\pm$ and at last, cubic terms enter when partial integration of the δ' terms in (4.33) leads to the appearance of the connection L_1 again.

To give an idea of the calculation we show the cancellation of the cubic terms proportional to $\psi_{2\,\pm}\psi_{2\,\pm}\chi_\pm$ in (4.33). According to (4.21) and (4.22) we have

$$\Big\{L_1(\gamma)\,,\chi^{\dot A}_\pm\Big\} \;=\; -\frac{\gamma}{2\rho(1\pm\gamma)^2}\,\Gamma^{IJ}_{\dot A\dot B}\chi^{\dot B}_\pm X^{IJ}\,\delta(z-z') + \frac{4\mathrm{i}\gamma^2}{\rho(1\pm\gamma)^2(1-\gamma^2)}\,\Gamma^K_{A\dot A}\psi^K_{2\,\pm}Y^A\,\delta(z-z')\;,$$

such that the cubic term $\psi_{2\,\pm}\chi_\pm\chi^{\dot B}_\pm$ from (4.24) gives the contribution

$$\big\{L_1(\gamma)\,,\mathrm{i}\,\rho\psi^K_{2\,\pm}\chi_\pm\Gamma^{IK}\chi_\pm\big\} \;\to\; \frac{-8\mathrm{i}\gamma^2}{\rho(1\pm\gamma)^2(1-\gamma^2)}\,\Gamma^{IM}_{\dot A\dot B}\Gamma^N_{A\dot B}\psi^M_{2\,\pm}\psi^N_{2\,\pm}\chi^{\dot A}Y^A\;, \tag{4.34}$$

to the r.h.s of (4.33). Next, there comes a contribution from the bracket between $P_\pm$ in $L_1(\gamma)$ and the $\rho\chi_\pm P_\pm$ part of the supersymmetry constraint (4.24), which is due to the quadratic fermionic terms in (4.19) and reads

$$\Big\{L_1(\gamma)\,,\pm 2\rho\chi^{\dot A}_\pm\Gamma^I_{A\dot A}P^A_\pm\Big\} \;\to\; \frac{-8\gamma^2}{\rho(1\pm\gamma)^2(1-\gamma^2)}\,\Gamma^{MN}_{AB}\,\Gamma^I_{B\dot A}\psi^M_{2\,\pm}\psi^N_{2\,\pm}\chi^{\dot A}Y^A\;. \tag{4.35}$$

Making use of (4.2) the two terms (4.34) and (4.35) sum up to

$$\frac{8\mathrm{i}\gamma^2}{(1\pm\gamma)^2(1-\gamma^2)}\,\Gamma^N_{A\dot A}\;\psi^I_{2\,\pm}\psi^N_{2\,\pm}\chi^{\dot A}Y^A\;. \tag{4.36}$$

Several further relevant terms arise from the Poisson brackets involving the $\rho\psi_{2\,\pm}\partial_\pm\sigma$ term in (4.24). Namely, $\{L_1(\gamma),\partial_0\sigma\}$ gives rise to several bilinear fermionic terms due to the brackets (4.20), (4.22) and eventually also due to

$$\{\gamma(z)\,,\partial_\pm\sigma(z')\} \;=\; -\frac{\gamma(1\mp\gamma)}{2\rho(1\pm\gamma)}\,\delta(z-z')\;.$$

Altogether they sum up to

$$\big\{L_1(\gamma)\,,2\rho\partial_\pm\sigma\;\psi^I_{2\,\pm}\big\} \;\to\; \frac{16\mathrm{i}\gamma^2(1\mp 4\gamma+\gamma^2)}{1\pm\gamma)^4(1-\gamma^2)}\,\Gamma^N_{A\dot A}\psi^I_{2\,\pm}\psi^N_{2\,\pm}\chi^{\dot A}Y^A\;. \tag{4.37}$$

Finally, the integrand of (4.33) has terms proportional to $\partial_z\delta(z-z')$ due to

$$\Big\{L_1(t)\,,\pm 2\rho\chi^{\dot A}_\pm\Gamma^I_{A\dot A}P^A_\pm\Big\} \;\to\; \pm\frac{1\mp\gamma}{1\pm\gamma}\,\Gamma^I_{A\dot A}\chi^{\dot A}_\pm Y^A\,\partial_z\delta(z-z')\;,$$

and

$$\left\{L_1(t)\,,\mp 2\rho\partial_1\psi^I_{2\,\pm}\right\} \;\to\; \pm\frac{4\gamma}{(1\pm\gamma)^2}\;\psi^K_{2\,\pm}X^{KI}\,\partial_z\delta(z-z')\;.$$

Upon partial integration in (4.33) and using (4.30) they give rise to

$$\mp\frac{1\mp\gamma}{2(1\pm\gamma)}\;\widehat{Q}^{KL}_{\pm}(\gamma)\Gamma^I_{A\dot{A}}\chi^{\dot{A}}_{\pm}\left[X^{KL},Y^A\right] \;\to\; \frac{8\mathrm{i}\gamma^2(1\mp\gamma)}{(1\pm\gamma)^5}\;\Gamma^I_{B\dot{A}}\Gamma^{MN}_{AB}\psi^M_{2\,\pm}\psi^N_{2\,\pm}\chi^{\dot{A}}Y^A\;,$$

and

$$\mp\frac{2\gamma}{(1\pm\gamma)^2}\;\widehat{P}^A_{\pm}(\gamma)\psi^K_{2\,\pm}\;\left[Y^A,X^{KI}\right] \;\to\; \frac{8\mathrm{i}\gamma^2(1\mp\gamma)}{(1\pm\gamma)^5}\;\Gamma^{MI}_{AB}\Gamma^N_{B\dot{A}}\psi^M_{2\,\pm}\psi^N_{2\,\pm}\chi^{\dot{A}}Y^A\;.$$

The sum of these two terms yields (again involving some Γ-matrix algebra (4.2))

$$\frac{-24\mathrm{i}\gamma^2(1\mp\gamma)}{(1\pm\gamma)^5}\;\Gamma^N_{A\dot{A}}\psi^I_{2\,\pm}\psi^N_{2\,\pm}\chi^{\dot{A}}Y^A\;. \tag{4.38}$$

Adding the different terms (4.36), (4.37) and (4.38) finally leads to

$$\frac{8\mathrm{i}\gamma^2}{(1\pm\gamma)^2(1-\gamma^2)}+\frac{16\mathrm{i}\gamma^2(1\mp 4\gamma+\gamma^2)}{(1\pm\gamma)^4(1-\gamma^2)}-\frac{24\mathrm{i}\gamma^2(1\mp\gamma)}{(1\pm\gamma)^5}\;=\;0\;. \tag{4.39}$$

We see, how the terms of the type $\psi_{2\,\pm}\psi_{2\,\pm}\chi_{\pm}$ from all the different sources eventually cancel. In a similar way all cubic fermionic terms in (4.33) can be shown to drop out. There remain only those contributions which transform "homogenously" under the transition matrix, i.e. which appear in the first line of the r.h.s. in (4.31).

Poisson algebra of charges

Eventually, we compute the Poisson algebra of the conserved charges that we have obtained above. As it turns out, it is completely sufficient to compute the Poisson brackets of the connection of the linear system (4.30). Namely, the result below coincides with (3.31) obtained above in the bosonic sector (i.e. setting all fermions to zero, whereby (4.30) reduces to the linear system (3.1).

A lengthy calculation gives the following Poisson brackets for the components of the linear system

$$\begin{aligned}
\left\{\widehat{Q}^{IJ}_1(\gamma_1),\widehat{Q}^{KL}_1(\gamma_2)\right\} &= \frac{2\gamma_1\gamma_2\,f^{IJ,KL}{}_{MN}}{\rho(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\;\left(\widehat{Q}^{MN}(\gamma_1)-\widehat{Q}^{MN}(\gamma_2)\right)\delta(x-y)\,,\\
\left\{\widehat{Q}^{IJ}_1(\gamma_1),\widehat{P}^{A}_1(\gamma_2)\right\} &= -\frac{\gamma_2^2(1-\gamma_1^2)}{\rho(1-\gamma_2^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\;\Gamma^{IJ}_{AB}\widehat{P}^B_1(\gamma_1)\,\delta(x-y)\\
&\quad+\frac{\gamma_1\gamma_2}{\rho(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\;\Gamma^{IJ}_{AB}\widehat{P}^B(\gamma_2)\,\delta(x-y)\;,\\
\left\{\widehat{P}^{A}_1(\gamma_1),\widehat{P}^{B}_1(\gamma_2)\right\} &= \frac{(1-\gamma_1^2)\gamma_2^2}{2\rho(1-\gamma_2^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\;\Gamma^{IJ}_{AB}\widehat{Q}^{IJ}_1(\gamma_1)\,\delta(x-y)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{(1-\gamma_2^2)\gamma_1^2}{2\rho(1-\gamma_1^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\,\Gamma^{IJ}_{AB}\widehat{Q}_1^{IJ}(\gamma_2)\,\delta(x-y)\\
&- \frac{2\delta^{AB}}{(1-\gamma_1^2)(1-\gamma_2^2)}\left(\frac{\gamma_1(1+\gamma_2^2)}{\rho(x)}+\frac{\gamma_2(1+\gamma_1^2)}{\rho(y)}\right)\delta'(x-y)\\
&- \frac{2\gamma_1\gamma_2}{\rho^2(1-\gamma_1^2)(1-\gamma_2^2)}\,\Gamma^{IJ}_{AB}\,\Phi_{IJ}\,\delta(x-y)
\end{aligned}
$$

with $\gamma_1\equiv\gamma(x,v)$, $\gamma_2\equiv\gamma(y,w)$, and the structure constants $f^{IJ,KL}{}_{MN}$ of $\mathfrak{so}(16)$. Translating this back into tensor notation (2.56) we arrive at

$$
\begin{aligned}
\left\{\overset{1}{L}_1(\gamma_1)\;,\;\overset{2}{L}_1(\gamma_2)\right\} \;=\; &-\frac{2\gamma_1\gamma_2}{\rho(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\left[\Omega_{\mathfrak{h}}\;,\;\overset{1}{L}_1(\gamma_1)+\overset{2}{L}_1(\gamma_2)\right]\delta(x-y)\\
&-\frac{2\gamma_2^2(1-\gamma_1^2)}{\rho(1-\gamma_2^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\left[\Omega_{\mathfrak{k}}\;,\;\overset{1}{L}_1(\gamma_1)\right]\delta(x-y)\\
&-\frac{2\gamma_1^2(1-\gamma_2^2)}{\rho(1-\gamma_1^2)(\gamma_1-\gamma_2)(1-\gamma_1\gamma_2)}\left[\Omega_{\mathfrak{k}}\;,\;\overset{2}{L}_1(\gamma_2)\right]\delta(x-y)\\
&-\frac{2\,\Omega_{\mathfrak{k}}}{(1-\gamma_1^2)(1-\gamma_2^2)}\left(\frac{\gamma_1(1+\gamma_2^2)}{\rho(x)}+\frac{\gamma_2(1+\gamma_1^2)}{\rho(y)}\right)\delta'(x-y)\;,
\end{aligned}
$$

and find them to be identical with the Poisson brackets (3.31) obtained above. Thus, we have shown, that the integrable structure of the bosonic sector of this model completely extends to its maximal supersymmetric version. The resulting algebra of observables will be (3.60), (3.61) and (3.62) with E_8 valued matrices $U_\pm(w)$ and $\mathcal{M}(w)$, respectively.

In particular, the analysis of the symmetry structure from section 3.4 remains valid. With the generators (3.71) of the affine symmetry at hand, it is straightforward to compute their action on the fermionic fields, given by the Lie-Poisson action of the affine algebra $\mathfrak{e}_9$. Let us however mention an open problem about the supersymmetric version of these symmetries, that is their transitivity. Whereas in the bosonic sector under certain assumptions on the phase space we have directly seen that (3.71) generates the full phase space, it is a priori not clear to which extent this statement holds in the supersymmetric case. This question is essentially related to the completeness of the set of conserved charges, that has been answered affirmatively only in the bosonic sector so far. Maybe, the full answer to this question has to be postponed until a complete quantum model is at hand (see the discussion in [105]).

5 Quantization

So far, we have achieved a complete reformulation of the classical model (2.45) in terms of the transition matrices as new fundamental variables providing a complete set of integrals of motion. This formulation reveals integrability and the classical symmetries in a beautiful way. The goal in this chapter is to find the quantum algebra underlying the classical structure (3.56)–(3.62). We restrict to the model with algebra $\mathfrak{g}=\mathfrak{sl}(N)$. The particular case $\mathfrak{g}=\mathfrak{sl}(2)$ related to the two Killing vector field reduction of Einstein gravity described in section 2.1 is analyzed in further detail.

5.1 Quantum algebra

In this section, we present the algebra which upon quantization replaces the Poisson algebra (3.60), (3.61) and (3.62). An essential additional ingredient is the requirement that the generators of the quantum algebra must be compatible with some quantum version of the relation (3.57).

Let us recall the classical algebra of integrals of motion (3.60), (3.61) for $\mathfrak{g} = \mathfrak{sl}(N)$. The maximal compact subalgebra of $\mathfrak{g}$ is $\mathfrak{h} = \mathfrak{so}(N)$ and the involution τ is given by $\tau(\xi) = -\xi^T$. It is $\Omega_{\mathfrak{sl}(N)} = \Pi_N - \frac{1}{N} I$ with the $N^2 \times N^2$ permutation operator Π_N:

$$(\Pi_N)^{ab,cd} = \delta^{ad}\delta^{bc} \; .$$

Accordingly we define its twisted analogue Π_N^τ by

$$(\Pi_N^\tau)^{ab,cd} \;\equiv\; \left(-\Pi_N^{T_1} + \tfrac{2}{N} I\right)^{ab,cd} \;\equiv\; -\delta^{ac}\delta^{bd} + \tfrac{2}{N}\delta^{ab}\delta^{cd} \; .$$

The notation $\Pi_N^{T_1}$ here denotes transposition in one of the two spaces in which Π_N lives.

The Poisson algebra (3.60), (3.61) then takes the form:

$$\left\{ \overset{1}{U}_\pm(v) \, , \, \overset{2}{U}_\pm(w) \right\} \;=\; \left[\frac{\Pi_N}{v-w}, \overset{1}{U}_\pm(v) \, \overset{2}{U}_\pm(w) \right] \; , \tag{5.1}$$

$$\left\{ \overset{1}{U}_\pm(v) \, , \, \overset{2}{U}_\mp(w) \right\} \;=\; \frac{\Pi_N}{v-w} \, \overset{1}{U}_\pm(v) \, \overset{2}{U}_\mp(w) - \overset{1}{U}_\pm(v) \, \overset{2}{U}_\mp(w) \frac{\Pi_N^\tau}{v-w} \; , \tag{5.2}$$

The $U_\pm(w)$ are related by complex conjugation (3.56) and further restricted by the group property:

$$\det U_\pm(w) = 1 \; , \tag{5.3}$$

and the relation (3.57):

$$\mathcal{M}_{\rm BM}(w) = U_+(w)U_-^T(w) = U_-(w)U_+^T(w) = \mathcal{M}_{\rm BM}^T(w) \; . \tag{5.4}$$

Understanding the matrix entries of the $U_\pm(w)$ as classical phase space functions, quantization amounts to replacing (5.1), (5.2) by corresponding commutator relations of an $\hbar$-graded algebra, such that these relations are compatible with certain quantum analogues of (5.3) and (5.4). This problem admits the following essentially unique solution [77]:[11]

The quantization of the Poisson algebra (5.1)–(5.4) is given by the $*$-algebra generated by the matrix entries of $N \times N$ matrices $U_\pm(w)$ subject to the exchange relations

$$\begin{aligned} R(v-w)\, \overset{1}{U}_\pm(v)\, \overset{2}{U}_\pm(w) &= \overset{2}{U}_\pm(w)\, \overset{1}{U}_\pm(v) R(v-w) \;, & (5.5)\\ R(v-w-{\rm i}\hbar)\, \overset{1}{U}_-(v)\, \overset{2}{U}_+(w) &= \overset{2}{U}_+(w)\, \overset{1}{U}_-(v) R^\tau\left(v-w+\tfrac{2}{N}{\rm i}\hbar\right)\, \chi(v-w) \;, & (5.6) \end{aligned}$$

with

$$R(v) \equiv vI - {\rm i}\hbar\Pi_N \;,\;\; R^\tau(v) \equiv vI - {\rm i}\hbar\Pi_N^\tau \;,\;\; \chi(v) \equiv \frac{\Gamma\left(\frac{-{\rm i}\hbar-v}{N{\rm i}\hbar}\right)\Gamma\left(\frac{(N+2){\rm i}\hbar-v}{N{\rm i}\hbar}\right)}{\Gamma\left(\frac{-v}{N{\rm i}\hbar}\right)\Gamma\left(\frac{(N+1){\rm i}\hbar-v}{N{\rm i}\hbar}\right)} \;, \tag{5.7}$$

with the usual Γ*-function satisfying* $\Gamma(1)=1$, $\Gamma(x+1)=x\Gamma(x)$.

The condition of unit determinant (5.3) is replaced by the quantum determinant

$$\begin{aligned} 1 &= {\rm qdet} U_\pm(w) & (5.8)\\ &\equiv \sum_{\sigma\in\mathfrak{S}_N} {\rm sgn}(\sigma) U_\pm^{1\sigma(1)}(w-(N-1){\rm i}\hbar) U_\pm^{2\sigma(2)}(w-(N-2){\rm i}\hbar)\dots U_\pm^{N\sigma(N)}(w) \end{aligned}$$

and the quantum form of (5.4) is given by

$$\mathcal{M}_{\rm BM}(w) \;\equiv\; U_+(w)U_-^T(w) \;=\; \left(U_+(w)U_-^T(w)\right)^T \;=\; \mathcal{M}_{\rm BM}^T(w) \;, \tag{5.9}$$

where "^{T}" here simply refers to the transposition of the classical $N \times N$ matrices. The $*$-operation is defined by

$$U_+(w)^* \equiv U_-(\bar w) \;, \tag{5.10}$$

and builds a conjugate-linear anti-multiplicative automorphism of the algebra (5.5)–(5.9).

There are several things to note about the algebra (5.5)–(5.10) before we come to the proof.

- The algebra (5.5)–(5.9) is isomorphic under rescaling of $\hbar$ with positive real numbers. Namely, this is absorbed by a rescaling of the spectral parameter w. Negative or complex rescaling would violate the assumed holomorphy behavior of the $U_\pm(w)$ at least in the classical limit.[12] We will in the following set $\hbar=1$.

[11] For simplicity we use the same notation for the classical and the quantum operators.

[12] Upon quantization, the holomorphy behavior of the classical functions $U_\pm(w)$ translates into analyticity of the action of the corresponding operators in dependence of the parameter w. This analyticity however depends on the topology of the concrete representation space, which has not been fixed so far.

Depending on the sign of $\hbar$, there is hence a $\mathbb{Z}_2$-freedom in constructing (5.5)–(5.9). This corresponds to the symmetry $(+ \leftrightarrow -)$ of the classical Poisson algebra (5.1)–(5.4), which is obviously broken after quantization. This freedom might be fixed from the later requirement of the existence of unitary representations.

- The algebra (5.5), (5.6) is no Hopf algebra. This follows already from the absence of a trivial representation of (5.5), (5.6). Even stronger, due to the singularity structure of (5.6) this algebra admits no finite-dimensional representations.

- The essential new ingredient of (5.5), (5.6) is the appearance of the twist τ in the mixed relations which has already appeared in the classical Poisson algebra. It is basically this peculiarity which requires a new representation theory to be developed. (Unfortunately, the notion of "twist" has been introduced for several different concepts for quantum groups in general and even for the Yangians in particular.)

- The definition of the quantum determinant (5.8) is known from the $\mathfrak{sl}(N)$ Yangian [56, 80, 97]. It encodes the generators of the center of the algebras (5.5). Here, we must in addition ensure that $\mathrm{qdet} U_\pm(w)$ also lies in the center of the full algebra (5.5), (5.6). It is this requirement which uniquely fixes the factor $\chi(v-w)$ in (5.6).

- A central extension of the type appearing in the mixed exchange relations (5.6) (i.e. the shift of the argument in the quantum R-matrix) has been introduced for quantum affine algebras in [110] and explicitly for the Yangian double in [65, 54]. Here, its value is uniquely fixed from the requirement of compatibility with (5.9). From the abstract point of view, the central extension takes the critical value at which the antisymmetric part of $\mathcal{M}$ generates a two-sided ideal (cf. (5.16) below), i.e. any representation of the algebra (5.5), (5.6) factorizes over this ideal. A common shift of both arguments in the R-matrices of (5.6) may be absorbed by redefinition of $U_+(w)$ and (5.9), (5.10), introducing a relative shift in the latter.

  The normal (untwisted) Yangian double has a critical value of the central extension at which it possesses an infinite dimensional center [110]. As we shall discuss in the next chapter, for $N=2$ the algebra (5.5)–(5.8) is in fact isomorphic to the normal centrally extended Yangian double at this critical level.

- Recalling remark 3.5, Drinfeld's Yangian and its double are obtained from (5.5) by expanding $U_+(w)$ and $U_-(w)$ around $w = \infty$ and $w = 0$, respectively. This however does not match their holomorphy behavior in our model. Formally treating the algebra (5.5) only in terms of the generating functions $U_\pm(w)$ [40], we may however adopt most of the results concerning the Yangian to this case.

  In fact, for $N=2$, the algebra underlying (5.5) in our case is a degeneration $\mathcal{A}_\hbar(\mathfrak{sl}(2))$ of the scaling limit of the elliptic affine algebra $\mathcal{A}_{p,q}(\mathfrak{sl}(2))$ [44, 66]. Again, what is eventually needed is a modification of this algebra in accordance with the twist of (5.6).

- The symmetry property (5.9) together with definition (5.10) guarantees that the object $\mathcal{M}_{\mathrm{BM}}(w) \equiv U_+(w)U_-^T(w)$ is symmetric and invariant under the $*$-map. To be precise, as an $N\times N$ matrix it is symmetric, i.e.

$$\mathcal{M}^{ab}_{\rm BM}(w) = \mathcal{M}^{ba}_{\rm BM}(w) \;,$$

and the matrix entries are invariant under the $*$-operation

$$\mathcal{M}^{ab}_{\rm BM}(w) = \big(\mathcal{M}^{ab}_{\rm BM}(w)\big)^* \;,\quad \text{for } \; w\in\mathbb{R} \;.$$

In a unitary representation these matrix entries will thus form self-adjoint operators. Thus, $\mathcal{M}_{\rm BM}(w)$ is the natural quantum object that according to (3.59) underlies the original classical field on the symmetry axis. It satisfies closed exchange relations

$$R(v-w)\,\overset{1}{\mathcal{M}}_{\rm BM}(v)R^\tau\big(w-v+(1+\tfrac{2}{N})\mathrm{i}\big)\,\overset{2}{\mathcal{M}}_{\rm BM}(w) \tag{5.11}$$

$$= \overset{2}{\mathcal{M}}_{\rm BM}(w)R^\tau\big(v-w+(1+\tfrac{2}{N})\mathrm{i}\big)\,\overset{1}{\mathcal{M}}_{\rm BM}(v)R(w-v)\;\frac{\chi(v-w)}{\chi(w-v)}\;,$$

which are obtained from (5.5), (5.6) and may be viewed as the quantization of (3.62).

The rest of this section is devoted to the proof of consistency of (5.5)–(5.10).

Associativity Denote by $Y_\pm$ the algebra generated by the the matrix entries of $U_\pm(w)$, respectively, with exchange relations (5.5). These are two copies of the well-known Yangian algebra [27] which provides the unique quantization of the Poisson algebra given by (5.1). Compatibility with associativity is equivalent to the Yang-Baxter equation

$$R_{12}(u-v)R_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R_{13}(u-w)R_{12}(u-v)\;, \tag{5.12}$$

for the quantum R-matrices R_{ij}, where the indices i,j denote the two spaces in which R_{ij} acts nontrivially.

Associativity of the full algebra (5.5), (5.6) is ensured by a modified (twisted) Yang-Baxter equation for R^τ:

$$R^\tau_{12}(u-v)R^\tau_{13}(u-w)R_{23}(v-w) = R_{23}(v-w)R^\tau_{13}(u-w)R^\tau_{12}(u-v)\;. \tag{5.13}$$

Validity of the classical version of this equation (i.e. modulo terms in $\hbar^3$) is a consequence of the fact, that τ is an algebra automorphism of $\mathfrak{g}$. For the quantum R-matrices R and R^τ in (5.7), the twisted Yang-Baxter equation (5.13) follows from

$$R^\tau\big(v+\tfrac{2}{N}\mathrm{i}\big) \;=\; -R^{T_1}(-v)\;, \tag{5.14}$$

and (5.12) by applying transposition and a shift of the argument in the first space.

Thus, whereas the exchange relations for $Y_\pm$ are uniquely given by (5.5) [27], for the mixed exchange relations (5.6) we may take the general ansatz

$$R(v-w+c_1\mathrm{i})\,\overset{1}{U}_-(v)\,\overset{2}{U}_+(w) \;= \overset{2}{U}_+(w)\,\overset{1}{U}_-(v)R^\tau(v-w+c_2\mathrm{i})\,\chi(v-w)\;. \tag{5.15}$$

Central extension The resulting algebra must respect the symmetry (5.9) of $\mathcal{M}(w)$. More precisely we demand the following: Denote the set of antisymmetric matrix entries of $\mathcal{M}_{\rm BM}$ by $\mathcal{I} \subset \mathcal{U}(Y_+ \oplus Y_-)$. Then we require that $\mathcal{I}$ spans a two-sided ideal in the sense that

$$(Y_+ \oplus Y_-)\,\mathcal{I} = \mathcal{I}\,(Y_+ \oplus Y_-)\ . \tag{5.16}$$

This relation ensures that the antisymmetry of $\mathcal{M}_{\rm BM}$ may be consistently imposed without inducing any further relations, i.e. any representation of $Y_\pm$ factorizes over $\mathcal{I}$. Equation (5.16) is not influenced by the choice of χ but uniquely determines the values of the parameters c_j in (5.15) to be

$$c_1 = -1\ , \quad c_2 = \tfrac{2}{N}\ .$$

This may be verified straight-forwardly e.g. by evaluating (5.15) and (5.9) in matrix components. At these values of the c_j the exchange relations between $U_\pm$ and $\mathcal{M}_{\rm BM}$ take the closed form

$$\begin{aligned} \chi(w-v)R(v-w)\, \overset{1}{U}_+(v)\, \overset{2}{\mathcal{M}}_{\rm BM}(w) &= \overset{2}{\mathcal{M}}_{\rm BM}(w) R^\tau\big(v-w+(1+\tfrac{2}{N}){\rm i}\big)\, \overset{1}{U}_+(v)\ , \\ R(v-w-{\rm i})\, \overset{1}{U}_-(v)\, \overset{2}{\mathcal{M}}_{\rm BM}(w) &= \overset{2}{\mathcal{M}}_{\rm BM}(w) R^\tau\big(v-w+\tfrac{2}{N}{\rm i}\big)\, \overset{1}{U}_-(v)\, \chi(v-w)\ , \end{aligned} \tag{5.17}$$

and indeed imply (5.16). These relations provide a quantization of (3.70) and shall play an important role for the quantum symmetries.

Quantum determinants The factor $\chi(v)$ in (5.6) is finally fixed from the requirement that the quantum determinants from (5.8) commute with everything such that the relations (5.8) are consistent with the algebra multiplication. It is known [56, 80] that the ${\rm qdet}U_\pm$ span the center of $Y_\pm$ respectively, thus $\chi(v)$ must ensure that they also commute with $Y_\mp$

$$[{\rm qdet}U_\pm(v)\,, Y_\mp] \;=\; 0\ . \tag{5.18}$$

Commutativity of ${\rm qdet}U_\pm$ with $Y_\pm$ essentially follows from the relation [97]

$$\begin{aligned} {\rm qdet}U_\pm(w)A_N &= A_N\, \overset{1}{U}_\pm(w)\, \overset{2}{U}_\pm(w-{\rm i}) \dots \overset{N}{U}_\pm(w-(N-1){\rm i}) \\ &= \overset{N}{U}_\pm(w-(N-1){\rm i}) \dots \overset{2}{U}_\pm(w-{\rm i})\, \overset{1}{U}_\pm(w) A_N\ , \end{aligned}$$

where A_N denotes the antisymmetrizer in the N auxiliary spaces. Upon successive use of the exchange relations (5.5) this leads to

$$\begin{aligned} & A_N R_{01} \dots R_{0N} A_N\, \overset{0}{U}_\pm(v)\, {\rm qdet}U_\pm(w) A_N \\ &\qquad\qquad = {\rm qdet}U_\pm(w) A_N\, \overset{0}{U}_\pm(v) A_N R_{01} \dots R_{0N} A_N\ , \end{aligned}$$

with

$$R_{0k} \equiv R_{0k}(v-w+(k-1){\rm i})\ .$$

With the additional relation

$$A_N R_{01} \dots R_{0N} A_N = A_N R_{0N} \dots R_{01} \; = \; \frac{v-w-\mathrm{i}}{v-w} \prod_{k=1}^{N} (v-w+(k-1)\mathrm{i})\, A_N \; , \quad (5.19)$$

it follows immediately, that $\mathrm{qdet}U_\pm$ commutes with all matrix entries of $U_\pm$. The factor on the r.h.s in (5.19) is most conveniently obtained from evaluating both sides on the particular vector $e_1 \otimes e_1 \otimes e_2 \otimes \dots \otimes e_N$.

In a similar way, the mixed relations (5.6) eventually yield

$$\begin{aligned} A_N R'_{01} \dots R'_{0N} A_N \overset{0}{U_-}(v)\, \mathrm{qdet}U_+(w) A_N & \\ = \; \mathrm{qdet}U_+(w) A_N \overset{0}{U_-}(v) A_N R''_{01} \dots R''_{0N} A_N \; , \end{aligned}$$

with

$$R'_{0k} \equiv R_{0k}(v-w+(k-2)\mathrm{i}) \; , \quad R''_{0k} \equiv R^{\tau}_{0k}(v-w+(k+\tfrac{2}{N}-1)\mathrm{i})\, \chi(v-w+(k-1)\mathrm{i}) \; .$$

From (5.19) we now obtain

$$A_N R'_{01} \dots R'_{0N} A_N = \frac{v-w-2\mathrm{i}}{v-w-\mathrm{i}} \prod_{k=1}^{N} (v-w+(k-2)\mathrm{i})\, A_N \; ,$$

as well as (cf. (5.14))

$$A_N R''_{01} \dots R''_{0N} A_N = \frac{w-v-N\mathrm{i}}{w-v-(N-1)\mathrm{i}} \prod_{k=1}^{N} (v-w+(k-1)\mathrm{i}) \chi(v-w+(k-1)\mathrm{i})\, A_N \; .$$

Combining these equations shows that (5.18) implies the functional equation

$$\prod_{k=1}^{N} \chi(v+k\mathrm{i}) \; = \; \frac{v-\mathrm{i}}{v+(N+1)\mathrm{i}} \; , \quad (5.20)$$

for $\chi(v)$. Existence and uniqueness of the solution of this equation follows from the expansion in the limit $v \to -\mathrm{i}\infty$ (corresponding to $\hbar \to 0$ with the condition that $\Im v < 0$), where the first coefficient is normalized according to

$$\chi(v) \; \sim \; 1 - \tfrac{\mathrm{i}}{v} \left(1 + \tfrac{2}{N}\right) + \mathcal{O}\left(\tfrac{1}{v^2}\right) \; , \qquad \text{for} \quad v \to -\mathrm{i}\infty \; ,$$

in order to obtain the correct classical limit (5.2) from (5.6). The function χ given in (5.7) indeed is the unique solution of (5.20) with this normalization.

The $*$-structure It remains to check that the $*$-operation defined by (5.10) is a conjugate-linear anti-multiplicative automorphism of the structure (5.5)–(5.9). Compatibility of (5.5) and (5.6) with (5.10) obviously follows from $R(\bar{u}) = -\overline{R(-u)}$, $R^\tau(\bar{u}) = -\overline{R^\tau(-u)}$, $\chi(\bar{u}) = \overline{\chi(u)}$ and the fact that R and R^τ are symmetric under permutation of the two spaces. Invariance of the restriction of unit quantum determinant (5.8) under the $*$-map follows from

$$\begin{aligned} \mathrm{qdet}(U_\pm(w))^* &= \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sgn}(\sigma) U_\mp^{N\sigma(N)}(\bar{w}) \dots U_\mp^{1\sigma(1)}(\bar{w}+(N-1)\mathrm{i}) \\ &= \mathrm{qdet}(U_\mp(\bar{w}+(N-1)\mathrm{i})) , \end{aligned}$$

where for the second identity we have employed one of the many properties of the quantum determinant [97]. Finally, compatibility of the symmetry relation (5.9) with the $*$-map follows directly from invariance of this relation under $*$:

$$\big(U_+(w)U_-^T(w)\big)^* = \big(U_+(w)U_-^T(w)\big)^T = U_+(w)U_-^T(w) \qquad \text{for } w \in \mathbb{R} .$$

This finishes the proof of consistency of (5.5)–(5.10).

5.2 $\mathfrak{g} = \mathfrak{sl}(2)$

To further illustrate the formulas of the preceding section, we will now discuss the particular case $\mathfrak{g} = \mathfrak{sl}(2)$. This is the model which we have described in detail in section 2.1 in the context of the two Killing vector field reduction of pure $4d$ Einstein gravity. It deserves interest as a midi-superspace model for quantum gravity; the corresponding quantum model has been introduced in [76].

There are several reasons, why the case $N=2$ is somewhat distinguished and simpler to treat compared to higher N. E.g. the involution τ is an inner automorphism of $\mathfrak{sl}(2)$.[13] Remarkably, this leads to an algebra isomorphism between our twisted and the normal Yangian double, however this is no $*$-algebra isomorphism.

The exchange relations (5.5), (5.6) for $N=2$ read

$$R(v-w) \overset{1}{U}_\pm(v) \overset{2}{U}_\pm(w) = \overset{2}{U}_\pm(w) \overset{1}{U}_\pm(v) R(v-w) , \tag{5.21}$$

$$R(v-w-\mathrm{i}) \overset{1}{U}_-(v) \overset{2}{U}_+(w) = \overset{2}{U}_+(w) \overset{1}{U}_-(v) R^\tau(v-w+\mathrm{i})\, \chi(v-w) , \tag{5.22}$$

with R and R^τ from (5.7), where the permutation operator Π and its twisted analogue Π^τ are explicitly given by

$$\Pi \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad \Pi^\tau \equiv I - \Pi^{T_1} \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} .$$

Moreover, the function χ may be evaluated from (5.7) and shrinks down to

$$\chi(v) = \frac{v(v-2\mathrm{i})}{(v-\mathrm{i})(v+\mathrm{i})} .$$

[13] In contrast, for $N>2$ the involution $\tau(\xi) = -\xi^T$ is the outer automorphism of $\mathfrak{sl}(N)$ which corresponds to reflection of the Dynkin diagram.

The quantum determinant is given by

$$\begin{aligned} \mathrm{qdet} U_\pm(w) &\equiv U_\pm^{11}(w-\mathrm{i})U_\pm^{22}(w) - U_\pm^{12}(w-\mathrm{i})U_\pm^{21}(w) \\ &= U_\pm^{11}(w)U_\pm^{22}(w-\mathrm{i}) - U_\pm^{21}(w)U_\pm^{12}(w-\mathrm{i}) \;=\; 1 \;, \end{aligned} \tag{5.23}$$

and the matrix product

$$\mathcal{M}_{\mathrm{BM}}(w) \equiv U_+(w)U_-^T(w) = \mathcal{M}_{\mathrm{BM}}^T(w) \;, \tag{5.24}$$

is symmetric under transposition and satisfies (5.11).

As mentioned above, for $\mathfrak{g} = \mathfrak{sl}(2)$ the involution τ is an inner automorphism generated by conjugation with

$$\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;.$$

This allows to "retwist" the mixed relations (5.22) by the following transformation:

$$\mathcal{U}_+(w) \equiv U_+(w)\sigma_2 \;, \qquad \mathcal{U}_-(w) \equiv U_-(w) \;. \tag{5.25}$$

These retwisted generators satisfy the exchange relations of the normal Yangian double:

$$R(v-w)\, \overset{1}{\mathcal{U}}_\pm(v)\, \overset{2}{\mathcal{U}}_\pm(w) \;=\; \overset{2}{\mathcal{U}}_\pm(w)\, \overset{1}{\mathcal{U}}_\pm(v) R(v-w) \;, \tag{5.26}$$

$$R(v-w-\mathrm{i})\, \overset{1}{\mathcal{U}}_-(v)\, \overset{2}{\mathcal{U}}_+(w) \;=\; \overset{2}{\mathcal{U}}_+(w)\, \overset{1}{\mathcal{U}}_-(v) R(v-w+\mathrm{i})\, \chi(v-w) \;, \tag{5.27}$$

at the critical level $k = -2$. At this level the center of the Yangian double becomes infinite-dimensional and is generated by the trace of the quantum current [110]

$$L(w) \equiv \left[\mathcal{U}_+(w)\mathcal{U}_-^{-1}(w)\right] \;. \tag{5.28}$$

Evaluating this in terms of our matrix $\mathcal{M}_{\mathrm{BM}}(w)$ from (5.24) yields

$$\mathrm{tr}\, L(w) = \mathcal{M}_{\mathrm{BM}}^{12}(w) - \mathcal{M}_{\mathrm{BM}}^{21}(w) \;. \tag{5.29}$$

Recall that the central extension of our structure was precisely determined by the requirement (5.16). For $N = 2$ the subspace $\mathcal{I}$ is one-dimensional. An explicit calculation shows that beyond (5.16), $\mathcal{I}$ even lies in the center of the algebra (5.21)–(5.22). Here we see the complete agreement with the normal Yangian double at critical level. We have thus equivalence of the twisted structure (5.21)–(5.22) with the untwisted (5.26)–(5.27), however supplied with the somewhat peculiar $*$-structure:

$$\mathcal{U}_\pm(w)^* = \pm\,\mathcal{U}_\mp(\bar{w})\,\sigma_2 \;.$$

For higher N this equivalence does not hold. Neither is there an algebra isomorphism between (5.5), (5.6) and the normal Yangian double, nor does a center emerge at our critical level, rather criticality is expressed by (5.16).

Remark 5.1 For explicit calculations it is sometimes useful to express the exchange relations (5.21), (5.22) in matrix components $U_\pm^{ab}(w)$. The mixed relations (5.22) e.g. may equivalently be written as

$$\begin{aligned}\left(1-\frac{\mathrm{i}^2}{(v-w)^2}\right)U_-^{ab}(v)U_+^{cd}(w) \;&=\; \left(1-\frac{\mathrm{i}}{v-w}\right)U_+^{cd}(w)U_-^{ab}(v) \\ &+\frac{\mathrm{i}}{v-w}\Big(U_+^{ad}(w)U_-^{cb}(v)+\delta^{bd}\,U_+^{cm}(w)U_-^{am}(v)\Big) \\ &+\frac{\mathrm{i}^2}{(v-w)^2}\,\delta^{bd}\Big(U_+^{am}(w)U_-^{cm}(v)-U_+^{cm}(w)U_-^{am}(v)\Big)\;.\end{aligned} \tag{5.30}$$

Interpreting the matrix entries of the $U_\pm$ as creation and annihilation operators, respectively, the r.h.s. of (5.30) can be viewed as sort of normal ordering [76].

5.3 Representations and symmetries

In this section, we touch the question of representations of the algebra (5.21)–(5.24) that has been obtained for $\mathfrak{g}=\mathfrak{sl}(2)$. First, we note, that (5.21), (5.22) admit no evaluation representations of the type the normal Yangian (5.5) does [81, 19]. Replacing $U_\pm(w)$ by R-matrices involving an additional (quantum) space, does not give a representation of (5.21), (5.22), since by no combination of R and R^τ for $U_\pm$, (5.22) can be traced back to the twisted Yang-Baxter equation (5.13). We have already mentioned above the absence of finite-dimensional representations of (5.5), (5.6).

Recall the abelian sector of the theory. In terms of the operators $A_\pm$ from (3.92), there is a canonical Fock space representation (3.99). Classically, the embedding of these variables into the full nonabelian model is obtained via exponentiation

$$U_\pm^{11}(w) \equiv \exp\int_0^\infty dk\, A_\pm(k)e^{\pm \mathrm{i}kw}\;. \tag{5.31}$$

Having quantized the abelian model, we may translate (5.31) back as an operator in (3.99) and for illustration study its action on the vacuum $|0\rangle$. Whereas $U_-^{11}(w)$ leaves the vacuum invariant, $U_+^{11}(w)$ creates a coherent state corresponding to the classical field which on the axis $x=0$ is peaked as a δ-function around $t=-w$. One may speculate, that similar representations are relevant for the algebra (5.21), (5.22).

A general class of representations is obtained from the following construction. Let V be a finite-dimensional representation of the Yangian algebra Y_- of (5.21) (generated by the $U_-(w)$). A representation of the full algebra (5.21)–(5.24) is then given by the space

$$\mathcal{U}(Y_+)V\Big/\,\mathcal{U}(Y_+)\Big(\mathcal{I}\,V\;\oplus\;(\mathrm{qdet}U_+(w)-\mathrm{id})\,V\;\oplus\;(\mathrm{qdet}U_-(w)-\mathrm{id})\,V\Big)\;, \tag{5.32}$$

where we start from the regular representation of $\mathcal{U}(Y_+)$ and subsequently divide out the relations (5.23) and (5.24). The action of Y_- on (5.32) is obtained from the exchange relations (5.22) (i.e. explicitly from (5.30)) and the defining action from Y_- on V. The fact that $\mathcal{U}(Y_+)\,\mathcal{I}\,V$ and $\mathcal{U}(Y_+)\,(\mathrm{qdet}U_\pm(w)-\mathrm{id})\,V$ are representations of Y_- is merely a consequence of (5.18) and (5.16), i.e. valid for any N. For the trivial representation $V_0=\mathbb{C}$, the representation (5.32) has the form of a direct generalization of (3.99).

To proceed with this class of representations, there are essentially three points to clarify:

- What are the finite-dimensional representations of Y_{-}?
- Is the representation (5.32) irreducible or does it contain irreducible parts?
- Is the representation (5.32) unitary with respect to the $*$-structure given in (5.10)?

At least the first point can be answered completely, the finite-dimensional representations of the Yangian are classified by highest weights. Even more explicit results are known for the special case $\mathfrak{g} = \mathfrak{sl}(2)$ [19]. All finite-dimensional irreducible representation are generated by evaluation representations. The latter are obtained from evaluating the quantum R-matrix from (5.7) on the tensor product of a (classical) two-dimensional vector space and an irreducible representation of $\mathfrak{su}(2)$ [81]. To be precise, these representations are labeled by an insertion point z and the dimension $r+1$ of the representation of $\mathfrak{su}(2)$; the action of $U_{-}(w)$ on a basis $v_0, \ldots, v_r$ is given by

$$U_{-}(w)\, v_k \;\; \equiv \;\; f(w-z;r) \begin{pmatrix} w-z-\frac{1}{2}(2r-k)\, v_k & (r-k+1)\, v_{k-1} \\ (k+1)\, v_{k+1} & w-z+\frac{1}{2}(2r-k)\, v_k \end{pmatrix} , \tag{5.33}$$

where we have set $v_{-1} \equiv v_{r+1} \equiv 0$. The factor $f(w-z;r)$ is chosen such that it ensures the relation (5.23); it may be expressed in terms of Γ-functions. We denote this representation by $V_z^{(r)}$. The action of Y_{-} on the tensor product $V_z^{(r)} \otimes V_y^{(s)}$ is given by the Hopf algebra structure of the Yangian [27]:

$$U_{-}^{ab}(w) \;\; (v_k \otimes v_l) \;\; = \;\; U_{-}^{am}(w) v_k \otimes U_{-}^{mb}(w) v_l \; , \quad \text{for} \;\; v_k \otimes v_l \in V_z^{(r)} \otimes V_y^{(s)} \tag{5.34}$$

Remark 5.2 The general formulas (5.17) evaluated for $N=2$ yield

$$\overset{1}{U}_{\pm}(v\mp\tfrac{1}{2}\mathrm{i}) \; \overset{2}{\mathcal{M}}_{\mathrm{BM}}(w) \; \overset{1}{U}{}_{\pm}^{-1}(v\mp\tfrac{1}{2}\mathrm{i}) \tag{5.35}$$

$$\begin{aligned} &= \;\; R^{-1}(v-w-\tfrac{1}{2}\mathrm{i}) \; \overset{2}{\mathcal{M}}_{\mathrm{BM}}(w) R^{\tau}\left(v-w+\tfrac{3}{2}\mathrm{i}\right) \chi(v-w+\tfrac{1}{2}\mathrm{i}) \\ &= \;\; \left(I + \frac{\mathrm{i}\Pi}{v-w-\frac{1}{2}\mathrm{i}}\right) \overset{2}{\mathcal{M}}_{\mathrm{BM}}(w) \left(I - \frac{\mathrm{i}\Pi^{\tau}}{v-w+\frac{3}{2}\mathrm{i}}\right) \; . \end{aligned}$$

It can be checked that this "adjoint" representation of $Y_{\pm}$ on the three symmetric matrix entries of $\mathcal{M}_{\mathrm{BM}}(w_0)$ coincides with the evaluation representation $V_{w_0}^{(2)}$ from (5.33).

The representation theory of the Yangian $Y(\mathfrak{sl}_2)$ is essentially contained in the following result [19]. Each finite-dimensional irreducible representation is isomorphic to a tensor product of evaluation representations. A finite tensor product

$$\bigotimes_{m=1}^{N} V_{z_m}^{(r_m)} \; , \tag{5.36}$$

is reducible iff there are m, n with $1 \leq m, n \leq N$ and j with $0<j\leq \min\,(r_m, r_n)$ such that

$$\pm\mathrm{i}(z_m - z_n) = \tfrac{1}{2}(r_m+r_n) - j + 1 \; . \tag{5.37}$$

The representations (5.32) are thus labeled by the tensor products (5.36). Restrictions on $V^{(N)}$ may arise from the requirement of "holomorphy" of the action of the $U_-(w)$. As discussed above the action of $V^{(N)}$ should depend analytically on w for $w \in H_-$, i.e. $\Im w < 0$.

We can further evaluate the structure of (5.32). Its simplest elements apart from $V^{(N)}$ are given by the "single excitations"

$$U_+^{ab}(w_0)\, v_k \;, \quad \text{with} \;\; v_k \in V^{(N)} \equiv \bigotimes_{m=1}^{N} V_{z_m}^{(r_m)} \;. \tag{5.38}$$

Obviously, they again form a representation of Y_-, namely $V_{w_0}^{(2)} \otimes V^{(N)}$. The precise embedding follows from

$$\begin{aligned} U_-^{ab}(w)\, \mathcal{M}_{\rm BM}(w_0)\, V^{(N)} &= \Big(U_-(w)^{an} \mathcal{M}_{\rm BM}(w_0)\, (U_-^{-1}(w))^{nm} \Big) U_-^{mb}(w)\, V^{(N)} \\ &\overset{(5.35)}{=} \Big(U_-(w)^{am}\, V_{w_0}^{(2)} \Big) \Big(U_-^{mb}(w)\, V^{(N)} \Big) \\ &\overset{(5.34)}{=} U_-^{ab}(w) \Big(V_{w_0}^{(2)} \otimes V^{(N)} \Big) \;, \end{aligned}$$

where $\mathcal{M}_{\rm BM}(w_0)\, V^{(N)}$ encodes a basis of linear combinations of (5.38).

According to the criterion (5.37), we see, that the vectors (5.38) for generic w_0 form an irreducible representation of Y_- again. In particular, this implies that via the relation (5.23) it is possible to obtain back all vectors from $V^{(N)}$ by further action of Y_- on (5.38). Thus, there is only a discrete set of vectors among (5.38) – with w_0 related to one of the z_m from (5.36) by (5.37) – that give rise to potential proper subrepresentations. It remains to study these vectors separately.

Having analyzed all vectors (5.38), one has almost the full information about irreducibility of the representation (5.32). This is due to the fact that "higher excitations"

$$U_+^{a_n b_n}(w_n) \;\ldots\; U_+^{a_0 b_0}(w_0)\, v_k \;,$$

are formal elements of the Y_- representation

$$V_{w_n}^{(2)} \otimes \ldots \otimes V_{w_0}^{(2)} \otimes V^{(N)} \;. \tag{5.39}$$

According to (5.37), irreducibility of (5.39) is equivalent to the irreducibility of the pairwise tensor products contained in (5.39) which reduces the analysis to (5.38).

In this way, the question of irreducibility of (5.32) can be answered. This may result in further relations to be divided out from (5.32) and/or lead to further restrictions on the basis representation $V^{(N)}$ from (5.36). The last question concerning unitarity constitutes a more serious problem. At present, it is not clear if under certain assumptions, (5.36) can be equipped with a scalar product such that it is compatible with (5.10) and (5.32) does not contain states of negative norm. Having outlined the general programme of studying the class of representations (5.32), we defer the full analysis to later investigations.

We close this section with a remark on the symmetry that may replace the Geroch group (3.71) upon quantization. It is known [11, 84] that Lie-Poisson symmetries of the type (3.69) are realized as adjoint representations of the corresponding quantum algebra. In our case this is precisely provided by the relations (5.35). Evaluating the r.h.s. leads to

$$\begin{aligned}
&\overset{1}{U}_{\pm}(v-\tfrac{1}{2}\mathrm{i}\mp\tfrac{1}{2}\mathrm{i})\,\overset{2}{\mathcal{M}}_{\mathrm{BM}}(w)\,\overset{1}{U}{}^{-1}_{\pm}(v-\tfrac{1}{2}\mathrm{i}\mp\tfrac{1}{2}\mathrm{i}) \qquad (5.40)\\
&\quad= \overset{2}{\mathcal{M}}_{\mathrm{BM}}(w)\left(1+\frac{\mathrm{i}}{v-w-\mathrm{i}}-\frac{\mathrm{i}}{v-w+\mathrm{i}}\right)\\
&\qquad+\frac{1}{2}\left(\Pi\,\overset{2}{\mathcal{M}}_{\mathrm{BM}}(w)-\overset{2}{\mathcal{M}}_{\mathrm{BM}}(w)\Pi^{\tau}\right)\left(\frac{\mathrm{i}}{v-w-\mathrm{i}}+\frac{\mathrm{i}}{v-w+\mathrm{i}}\right)\,.
\end{aligned}$$

This explicitly shows that after projecting the first space onto a $\mathfrak{g}$-valued function $\Lambda(v)$ the l.h.s. becomes

$$\overset{1}{\mathrm{tr}}\left(\overset{1}{\Lambda}(v)\left[\overset{1}{U}_{\pm}(v-\tfrac{1}{2}\mathrm{i}\mp\tfrac{1}{2}\mathrm{i})\,,\,\overset{2}{\mathcal{M}}_{\mathrm{BM}}(w)\right]\overset{1}{U}{}^{-1}_{\pm}(v-\tfrac{1}{2}\mathrm{i}\mp\tfrac{1}{2}\mathrm{i})\right)\,,$$

with classical limit (3.69). The r.h.s. correspondingly reduces to (3.70) with the singularity at $v=w$ "quantum split" into

$$\frac{1}{v-w}\;\rightarrow\;\frac{1}{2}\left(\frac{1}{v-w+\mathrm{i}}+\frac{1}{v-w-\mathrm{i}}\right)\,, \qquad (5.41)$$

where the shifts in the denominators are of order $\hbar$. This may give an indication of how to deform the integration path ℓ in (3.71) after quantization.

The picture obviously is far from being completed, however throughout this section we have obtained several hints which features we suspect to eventually face. Let us emphasize the repeated occurence of the discrete shifts in the w-plane – (5.23), (5.35), and (5.41). In the gravitational context, where according to (3.59) the spectral parameter plane acquires some space-time meaning, this may give rise to speculating about a natural arising of discrete nonlocal structures [76]. Another allusion in this direction comes from (3.48) which suggests to represent the conformal factor σ at spacelike infinity by supplying (5.21)–(5.24) with a derivative operator $\mathrm{i}\partial/\partial w$. Its exponential $\exp\sigma$ (related to the deficit angle and the matter Hamiltonian in $3d$ cylindrically symmetric gravity) then translates into a discrete step operator.

6 Isomonodromic Structures in Dimensionally Reduced Gravity

This chapter is somewhat decoupled from the rest of the thesis. Here, we present the so-called isomonodromic approach to the model of dimensionally reduced gravity (2.45), which has been initiated in [70, 71] and elaborated in [72, 74, 104]. One of the motivations of this programme was the seeming dead end of the canonical formalism with the nonultralocal Poisson brackets (2.57). With the results presented in the last chapters we have however carried out the canonical approach to a much further stage which also appears to naturally capture the classical symmetries of the model and thus to build a reliable basis for quantization.

Still, the isomonodromic approach bears several interesting features. First, in relation with the "two-time" Poisson structure to be introduced it is manifestly two-dimensional covariant. It allows application not only to Kaluza-Klein reduction of spatial dimensions but also to those involving the timelike dimension (including e.g. stationary axisymmetric solutions). Further highlights are the decoupling of the chiral halves in the deformation equations (i.e. commutativity of the two Hamiltonian flows), the quantum group structure of the algebra of observables and the link to (a modified version of) the Knizhnik-Zamolodchikov equations from conformal field theory, which arise in the role of the Wheeler-DeWitt equations here.

6.1 Hamiltonian description of isomonodromic deformations

In this section, we describe a multi-time Hamiltonian formulation of isomonodromic deformations of meromorphic connections on the Riemann sphere due to [58]. Quantization of this system naturally leads to the Knizhnik-Zamolodchikov system [68].

We consider the space of holomorphic Lie-algebra valued one-forms on the punctured Riemann sphere, that are meromorphic with simple poles on the whole sphere. These forms may be viewed as connections on a trivial bundle. Introducing local coordinates on the sphere by marking a point ∞, an element $A(\gamma)d\gamma$ of this space is uniquely determined by its poles γ_j and the corresponding residues A_j taking values in $\mathfrak{g}$:

$$A(\gamma) = \sum_{j=1}^{N} \frac{A_j}{\gamma - \gamma_j} \ . \tag{6.1}$$

Holomorphic behavior at infinity is ensured by

$$Q \equiv \sum_j A_j = 0 \ . \tag{6.2}$$

There is a natural Poisson structure on the space of holomorphic connections on the punctured complex plane, that may be formulated in the equivalent expressions:

$$\{A_i^A, A_j^B\} \;=\; \delta_{ij} f^{AB}{}_C\, A_i^C \;, \tag{6.3}$$

$$\Leftrightarrow \qquad \{A^A(\gamma), A^B(\mu)\} \;=\; -f^{AB}{}_C\, \frac{A^C(\gamma) - A^C(\mu)}{\gamma - \mu} \;, \tag{6.4}$$

$$\Leftrightarrow \qquad \left\{ \overset{1}{A}(\gamma), \overset{2}{A}(\mu) \right\} \;=\; \left[r(\gamma-\mu)\, ,\, \overset{1}{A}(\gamma) + \overset{2}{A}(\mu) \right] \;, \tag{6.5}$$

with the structure constants $f^{AB}{}_C$ of the algebra $\mathfrak{g}$ and a classical r-matrix $r(\gamma) = \gamma^{-1}\Omega_{\mathfrak{g}}$, where $\Omega_{\mathfrak{g}} = t^A \otimes t_A$ denotes the Casimir element of $\mathfrak{g}$.

The condition (6.2) that restricts the connection to live on the sphere, transforms as a first-class constraint under this bracket: $\{Q^A, Q^B\} = f^{AB}{}_C\, Q^C$.

Holomorphic bracket from gauge fixed Chern-Simons theory

The holomorphic bracket (6.3) is induced by holomorphic gauge fixing of the fundamental Atiyah-Bott symplectic structure. The first-class constraint (6.2) ensuring $A(\gamma)$ to live on the sphere, arises naturally as surviving flatness condition, generating the constant gauge transformations. Let us shortly describe this relation.

The space of smooth connections on a Riemann surface is endowed with the natural symplectic form [5]

$$\omega = \mathrm{tr} \int \delta \mathrm{A} \wedge \delta \mathrm{A} \;,$$

that gives the Poisson bracket

$$\left\{ A_\gamma^A(\gamma)\, , A_{\bar\gamma}^B(\mu) \right\} \;=\; \delta^{AB} \delta^{(2)}(\gamma - \mu) \;, \tag{6.6}$$

where the connection A is split into $A_\gamma d\gamma + A_{\bar\gamma} d\bar\gamma$ and the δ-function is understood as a real two-dimensional δ-function: $\delta^{(2)}(x+iy) \equiv \delta(x)\delta(y)$.

The condition of flatness is $F = dA + A \wedge A = 0$ and builds an algebra of first-class constraints

$$\{F^A(\gamma), F^B(\mu)\} = f^{AB}{}_C\, F^C(\gamma) \delta^{(2)}(\gamma - \mu) \;.$$

generating the gauge transformations

$$A \mapsto gAg^{-1} + dgg^{-1} \;. \tag{6.7}$$

These brackets and constraints arise naturally from the Chern-Simons action. They may be extended to punctured Riemann surfaces if the singularities of the connection restrict to first order poles, leading to δ-function-like singularities of the curvature.[118, 34]

In order to extend these structures to holomorphic connections, first the phase space has to be enlarged in a natural way from real connections in terms of which Chern-Simons theory is usually formulated, to one-forms that take values in the complexified Lie algebra, as the split halfs $A_\gamma d\gamma$ and $A_{\bar\gamma} d\bar\gamma$ described above already do. Then, also the gauge freedom (6.7) is enlarged to the corresponding complex gauge group. We fix this gauge freedom by choosing the gauge $A_{\bar\gamma}=0$ that makes flatness turn into holomorphy.

The bracket between constraints and gauge-fixing condition is of the form:

$$\begin{aligned}\{F^A(\gamma), A^B_{\bar\gamma}(\mu)\} &= -\delta^{AB}\partial_{\bar\gamma}\delta^{(2)}(\gamma-\mu) + f^{AB}{}_C\, A^C_{\bar\gamma}(\gamma)\,\delta^{(2)}(\gamma-\mu) \\ &\approx -\delta^{AB}\partial_{\bar\gamma}\delta^{(2)}(\gamma-\mu)\ .\end{aligned} \tag{6.8}$$

This matrix can be inverted using $\partial_{\bar\gamma}\frac{1}{\gamma} = -2\pi\mathrm{i}\delta^{(2)}(\gamma)$. With the standard Dirac procedure [26] one further obtains the holomorphic bracket (6.4) for the remaining variables $A_\gamma(\gamma)$ [43].

Note that because of the appearance of the derivative $\partial_{\bar\gamma}$ in (6.8), the holomorphic part of the constraints $F^A(\gamma)$ survives as a first-class constraint. Since holomorphic functions on the sphere are constants, this becomes

$$\int F^A(\gamma)d\gamma d\bar\gamma = \int \partial_{\bar\gamma}A^A(\gamma)d\gamma d\bar\gamma = \sum_j A^A_j = Q^A\ , \tag{6.9}$$

and generates the remaining gauge transformations (6.7) with constant g.

Hamiltonian formulation of isomonodromic deformation

We now describe isomonodromic deformation on the sphere in terms of the holomorphic Poisson structure. Consider the system of linear differential equations:

$$\partial_\gamma\Psi(\gamma) = A(\gamma)\Psi(\gamma)\ . \tag{6.10}$$

For definiteness we choose some matrix representation of $\mathfrak{g}$ on a vector space V_0, such that $\Psi(\gamma)$ accordingly takes values in the exponentiated representation of the associated Lie-group $\mathbf{G}$.

As $A(\gamma)$ has simple poles, the function $\Psi(\gamma)$ lives on a covering of the punctured sphere. Let Ψ be normalized to $\Psi(\infty) = I$, thereby marking one of the points ∞ on this covering. In the neighborhood of the points γ_i, the function Ψ is given by:

$$\Psi(\gamma) \ = \ G_i\Psi_i(\gamma)(\gamma-\gamma_i)^{T_i}C_i\ , \tag{6.11}$$

with $\Psi_i(\gamma) = I + \mathcal{O}(\gamma-\gamma_i)$ being holomorphic and invertible. The relation to the residues of the connection (6.1) is given by $A_i = G_iT_iG_i^{-1}$.

The local behavior (6.11) also yields explicit expressions for the monodromies around the singularities:

$$\Psi(\gamma) \mapsto \Psi(\gamma)M_i\ , \quad \text{for } \gamma \text{ encircling } \gamma_i\,, \text{ with } \ M_i = C_i^{-1}\exp(2\pi\mathrm{i}T_i)\,C_i\ .$$

Note that the normalization $\Psi(\infty) = I$ couples the freedom of r.h.s. multiplication in the linear system (6.10) to the left action of constant gauge transformations (6.7) on Ψ. Under (6.7) thus Ψ transforms as $\Psi\mapsto g\Psi g^{-1}$ implying $M_i\mapsto gM_ig^{-1}$.

The aim of isomonodromic deformation [59] is the investigation of a family of linear systems (6.10) parameterized by the choice of singular points γ_i, that have the same monodromies. In other words, one studies the change of the connection data A_i with respect to a change in the parameters of the Riemann surface that is required to keep the monodromy data constant. Treating $A(\gamma)$ and $\Psi(\gamma)$ as functions of γ and γ_i, these isomonodromy conditions impose a formal condition of γ_i-independence of the monodromy data T_i and C_i.

This requires that the function $\partial_i\Psi\Psi^{-1}(\gamma)$ has a simple pole in γ_i : [14]

$$\partial_i\Psi(\gamma) \;=\; \frac{-A_i}{\gamma-\gamma_i}\;\Psi(\gamma)\;. \tag{6.12}$$

Compatibility of these equations with the system (6.10) yields the classical Schlesinger equations [112]:

$$\partial_i A_j = \frac{[A_i,A_j]}{\gamma_i-\gamma_j}\;,\quad \text{for } j\neq i\;,\qquad \partial_i A_i = -\sum_{j\neq i}\frac{[A_i,A_j]}{\gamma_i-\gamma_j}\;. \tag{6.13}$$

A multi-time Hamiltonian description of this dependence has been given in [58] with the Hamiltonians

$$H_i = \sum_{j\neq i}\frac{\mathrm{tr}(A_iA_j)}{\gamma_i-\gamma_j}\;, \tag{6.14}$$

generating the commuting γ_i-flows (6.13) in the holomorphic Poisson-bracket (6.3), i.e.

$$\partial_i A_j = \{A_j,H_i\}\;,\qquad \{H_i,H_j\}=0\;. \tag{6.15}$$

The Poisson structure is interpreted as a multi-time structure in the sense that (6.3) is defined for the residues $A_j(\{\gamma_i\})$ at coinciding γ_i and translated to different γ_i by means of (6.15).

Quantization and Knizhnik-Zamolodchikov system

As was noticed by Reshetikhin [109], quantization of this system leads to the Knizhnik-Zamolodchikov equations, that are known as differential equations for correlation functions in conformal field theory [68].

Quantization is performed straightforwardly by replacing the Poisson structure (6.3) by commutators. Shifting the γ_i-dependence (6.13) of the operators A_i^A into the states on which these operators act corresponds to a transition from the Heisenberg picture to the Schrödinger picture in ordinary quantum mechanics. In the Schrödinger representation the quantum states $|\omega\rangle$ then are sections of a holomorphic $V^{(N)}\equiv\bigotimes_j V_j$ vector bundle over

$$X_0\equiv\mathbb{C}^N\setminus\{\text{diagonal hyperplanes}\}\;.$$

The γ_i-independent operator-valued coordinates of A_i are realized as

$$A_i^A = \mathrm{i}\hbar\; I\otimes\ldots\otimes t_i^A\otimes\ldots\otimes I \tag{6.16}$$

where t_i^A acts in the representation V_i. In this Schrödinger picture the quantum states $|\omega\rangle$ then obey the following multi-time γ_i-dynamics

$$\partial_i|\omega\rangle = H_i|\omega\rangle = \mathrm{i}\hbar\sum_{j\neq i}\frac{\Omega_{ij}}{\gamma_i-\gamma_j}\;|\omega\rangle \tag{6.17}$$

[14]The derivative ∂_i here and in the following denotes $\partial/\partial\gamma_i$, the derivative with respect to the position of the singularity γ_i.

Here, $\Omega_{ij} = \mathrm{tr}(t_i^A \otimes t_{j\,A})$ denotes the Casimir element $\Omega_{\mathfrak{g}}$ of the algebra $\mathfrak{g}$, acting on V_i and V_j. The system (6.17) defines horizontal sections on the bundle of quantum states and coincides with the famous Knizhnik-Zamolodchikov system [68].

Remark 6.1 System (6.17) may be equivalently rewritten in the Heisenberg picture introducing the multi-time evolution operator $U_N(\{\gamma_i\})$ (as the general solution of (6.17)) by

$$\partial_i U_N = H_i U_N \; , \quad U_N(\{\gamma_i = 0\}) = \mathrm{id} \; . \tag{6.18}$$

Then in terms of the variables $U_N A_i^A U_N^{-1}$ the quantum equations of motion give rise to higher-dimensional Schlesinger equations with the matrix entries A_i^a being operators in V. These equations turn out to be a very special case of the general $(\dim V_0 \times \dim V^{(N)})$-dimensional classical Schlesinger system.

6.2 Isomonodromic sector in dimensionally reduced gravity

In this section, we introduce new fundamental variables for the system of dimensionally reduced gravity studied in the previous chapters. In terms of the connection of the linear system (3.1), the equations of motion bear some resemblance with the deformation equations obtained in (6.13). This suggests to adopt the holomorphic Poisson structure (6.4) which leads to a two-time Hamiltonian formulation of dimensionally reduced gravity.

Starting from the linear system (3.1) we consider the object

$$\Psi(x,t,\gamma) \;\equiv\; \mathcal{V}(x,t)\, \tau\Big(\widehat{\mathcal{V}}^{-1}(x,t,\gamma)\Big) \; . \tag{6.19}$$

It satisfies the linear system

$$\partial_\pm \Psi \Psi^{-1} \;=\; \frac{2}{1 \pm \gamma}\, \mathcal{V} P_\pm \mathcal{V}^{-1} \;=\; \frac{1}{1 \pm \gamma}\, \partial_\pm M M^{-1} \; , \tag{6.20}$$

with the matrix M from (2.42). These linear differential equations have been the basis for the isomonodromic ansatz.

The main objects we are going to consider as fundamental variables in the sequel are certain components of the following $\mathfrak{g}$-valued one-form

$$\boldsymbol{A} \equiv \mathrm{d}\Psi \Psi^{-1} \tag{6.21}$$

In particular, we are interested in the components

$$\boldsymbol{A} = A_\gamma \mathrm{d}\gamma + A_+ \mathrm{d}x^+ + A_- \mathrm{d}x^- = A_w \mathrm{d}w + \tilde{A}_+ \mathrm{d}x^+ + \tilde{A}_- \mathrm{d}x^- \tag{6.22}$$

where $(\gamma, x^\pm)$ and $(w, x^\pm)$, respectively, are considered to be independent variables. The main object in the sequel will be the particular component A_γ for which we use the shortened notation $A \equiv A_\gamma$.

Moreover, we will restrict our study to that sector of the theory, where A is a single-valued meromorphic function of γ, i.e. that also $\boldsymbol{A}$ is single-valued and meromorphic in γ. A solution Ψ of (6.20) with this property is called *isomonodromic*, as its monodromies in the γ-plane then have no w-dependence due to (6.21). In fact, this sector of the theory already covers the most interesting physical solutions.

Further on, we immediately get the following relations:

$$\partial_\pm M M^{-1} \;=\; \mp 2\rho^{-1}\partial_\pm\rho\, A(x^\pm,\gamma)\Big|_{\gamma=\mp 1}\,, \tag{6.23}$$

as a corollary of (6.20) and (3.2). Moreover, the linear system (6.20) and definition (6.22) imply:

$$A_w = \frac{\partial\gamma}{\partial w}A\,,\quad \tilde{A}_\pm \;=\; 2\rho^{-1}\partial_\pm\rho\,\frac{A(\mp 1)}{1\pm\gamma}\,,\quad A_\pm = \rho^{-1}\partial_\pm\rho\,\frac{2A(\mp 1)-\gamma(1\mp\gamma)A(\gamma)}{1\pm\gamma}\,.$$

For asymptotically flat solutions of (2.22) the linear system (6.20) admits the normalization

$$\Psi(\gamma\!=\!\infty) = I, \tag{6.24}$$

which implies regularity of A at infinity:

$$A_\infty \equiv \lim_{\gamma\to\infty}\gamma A(\gamma) = 0 \tag{6.25}$$

The definition of $\boldsymbol{A}$ as pure gauge (6.21) implies integrability conditions on its components, which in particular give rise to the following closed system for $A(\gamma)$:

$$\partial_\pm A \;=\; [A_\pm\,, A\,] + \partial_\gamma A_\pm\,. \tag{6.26}$$

The main advantage of this system in comparison with the original equations of motion in terms of M (2.22) is, that the dependence on the coordinates $x^\pm$ is now completely decoupled. Once the system (6.26) is solved, it is easy to check that the equations (6.23) are compatible and the field M restored by means of them satisfies (2.22). This decoupling of x^+ and x^- allows to treat (6.26) in the framework of a manifestly covariant two-time Hamiltonian formalism, where the field $A(\gamma)$ is considered as the new basic object.

For this purpose we equip $A(\gamma)$ with the (equal-$x^\pm$) Poisson structure from (6.5):

$$\Big\{\overset{1}{A}(\gamma)\,,\overset{2}{A}(\mu)\Big\} \;=\; \Big[\frac{\Omega_{\mathfrak{g}}}{\gamma-\mu}\,,\,\overset{1}{A}(\gamma)+\overset{2}{A}(\mu)\Big]\,. \tag{6.27}$$

The relations

$$\big\{A(\gamma)\,,\,\rho^{-1}\partial_\pm\rho\,\mathrm{tr}A^2(\mp 1)\big\} \;=\; 2\,[A_\pm(\gamma)\,,\,A(\gamma)]\,, \tag{6.28}$$

compared with the equations of motion (6.26) give rise to defining the Hamiltonians

$$\mathcal{H}_\pm \equiv \tfrac{1}{2}\rho^{-1}\partial_\pm\rho\,\mathrm{tr}A^2(\mp 1)\,,\qquad \text{with}\quad \{\mathcal{H}_+,\mathcal{H}_-\} = 0\,. \tag{6.29}$$

We call the $x^\pm$-dynamics that is generated by $\mathcal{H}_\pm$ the *implicit* time dependence of the fields. The remaining $x^\pm$-dynamics is referred to as *explicit* time dependence.

In general, the variables $A(\gamma)$ themselves are explicitly time-dependent according to (6.26) and (6.28). The motivation for introducing (6.29) originates from [70], where it has been shown, that in essential sectors of the theory (simple pole singularities in the connection A), it is possible to identify a complete set of explicitly time-independent variables. Let us briefly recall this.

First order poles In this simplest case considered in [70, 71] we assume that $A(\gamma)$ has only simple poles, i.e.

$$A(\gamma) = \sum_{j=1}^{N} \frac{A_j(x^\pm)}{\gamma - \gamma_j} \,, \tag{6.30}$$

where according to (6.20) all γ_j satisfy (3.2), i.e. $\gamma_j = \gamma(x^\pm, w_j)$, $w_j \in \mathbb{C}$. Then the equations of motion (6.26) yield

$$\partial_\pm A_j \;=\; \rho^{-1}\partial_\pm\rho \sum_{k\neq j} \frac{[A_k,\; A_j]}{(1-\gamma_k)(1-\gamma_j)} \;=\; \sum_{k=1}^{N} \partial_\pm\gamma_k\, \partial_k A_j \;, \tag{6.31}$$

with the γ_k dependence from (6.13). The Poisson brackets (6.27) reduces to

$$\{A_i^A, A_j^B\} \;=\; \delta_{ij} f^{AB}{}_C\, A_j^C \;, \tag{6.32}$$

i.e. in this case, the residues A_j together with the set of (hidden constant) positions of the singularities $\{w_j\}$ give the full set of explicitly time-independent variables.

Comparing the equations of motion in this sector (6.31) with the isomonodromic deformation equations (6.13) suggests to understand the $x^\pm$ dependence of the residues as an isomonodromic dependence generated by the two Hamiltonians (6.29).

Remark 6.2 Introduction of the Poisson structure (6.27) has been motivated from the mathematical point of view by the similarity of the equations of motion (6.31) with the isomonodromic formalism described in the previous section. However, a priori this structure is not canonically derived from the original Lagrangian (2.45). Dimensionally reduced gravity allows an alternative Chern-Simons Lagrangian formulation [72], such that (6.27) may be obtained from (6.6) by holomorphic gauge fixing. An honest comparison to the canonical Poisson structure (2.50), (2.57) of (2.45) should be worked out on the space of observables, where due to spacetime-diffeomorphism invariance at least no principal difference between one- and two-time structures appears.

Due to the $\rho, \tilde{\rho}$ dependence, the singularities γ_i have become field dependent and thus exhibit explicit time-dependence in the sense of (6.29). In order to gain a complete Hamiltonian description, we additionally introduce the following Poisson brackets

$$\Big\{\rho^\pm, -\partial_\pm\sigma\Big\} = \partial_\pm\rho \;, \tag{6.33}$$

where $\rho^\pm$ refer to the decomposition of ρ into left- and right-movers (2.17).[15] The dynamics in $x^\pm$ directions then is completely given by the Hamiltonian constraints $\mathcal{C}_\pm$

$$\mathcal{C}_\pm \;\equiv\; -\partial_\pm\sigma + \rho^{-1}\partial_\pm\rho\,\mathrm{tr}A^2(\mp 1) \;=\; \frac{T_{\pm\pm}}{\partial_\pm\rho} \;\approx\; 0 \;. \tag{6.34}$$

I.e. for any functional F we have

$$\frac{dF}{dx^\pm} = \{F, \mathcal{C}_\pm\} \;. \tag{6.35}$$

[15] Despite their index the fields $\rho^\pm$ are obviously scalars under conformal transformations.

Remark 6.3 The Hamiltonian constraints (6.34) are obviously related to the conformal constraints (2.58), as also the Poisson structure (6.33) is certainly inspired by (2.50). The fact, that both, (6.34) and (2.58) differ by a factor of $\partial_\pm\rho$ from their "canonical ancestors" is related to the nature of the two-time Poisson structure, e.g. required by conformal covariance. The precise embedding of the two-time structure into the canonical formalism is still somewhat unclear. As remarked above, an honest comparison had to be performed on the space of observables.

Remark 6.4 The above reduction (6.31) of the original equations of motion shows a remarkable general feature: the number of dimensions has been effectively reduced from two to one. Recall that the initial values of the physical fields are usually given on a spacelike hypersurface, whereas their evolution in the time direction is described by the equations of motion. Here, on the contrary we have evolution equations for the time direction as well as for the space direction and the two flows commute. The knowledge of the initial values of $A(\gamma)$ at one space-time point is sufficient to reconstruct the whole solution by means of (6.26).

This may be understood as follows: the spatial dimension which previously provided the initial data has been traded for an additional dimension parametrized by the spectral parameter. In fact, given the spectral parameter current $A(\gamma)$ at fixed $\gamma = \pm 1$ on a spacelike hypersurface (which according to (6.23) are nothing but the original currents) allows us to evolve it in time by means of the equations of motion and into the γ-direction via the compatibility equations (6.26). Vice versa, given $A(\gamma)$ at fixed space-time point but for all γ one can deduce its space evolution from the compatibility equations.

The isomonodromic ansatz (6.30) is finally employed to parametrize the behavior of the spectral parameter current in the γ-plane by a discrete (even finite) set of variables, such that the original field theory reduces to an "N-particle" problem (localized in the spectral parameter plane). In this way we have arrived at an effectively one-dimensional description of the $2d$ theory without giving up the nontriviality of the solutions.

Higher order poles The isomonodromic framework allows natural generalization to that sector of the theory, where $A(\gamma)$ is assumed to be a meromorphic function of γ, which we shall present here. A further extension of this framework to the full phase space of arbitrary connections A, that is strongly inspired from the treatment of the simple pole case, has been sketched in [104].

Assume that $A(\gamma)$ has higher order poles in the γ-plane:

$$A(\gamma) = \sum_{j=1}^{N} \sum_{k=1}^{r_j} \frac{A_j^k(x^\pm)}{(\gamma - \gamma_j)^k} \,. \tag{6.36}$$

The Poisson structure (6.27) in terms of A_j^k has the form:

$$\left\{ (A_i^k)^A, (A_j^l)^B \right\} = \begin{cases} \delta_{ij} f^{AB}{}_C \, (A_j^{k+l-1})^C & \text{for } k+l-1 \leq r_j \\ 0 & \text{for } k+l-1 > r_j \end{cases} , \tag{6.37}$$

building a set of mutually commuting truncated half affine algebras.

However, it turns out that for $r_j > 1$ the variables A_j^k for $k=1,\dots r_j-1$ have non-trivial Poisson brackets with $\partial_\pm\sigma$, and, therefore, are not explicitly time-independent. The problem of identification of explicitly time-independent variables can be solved in the following way. Consider

$$A_w(\gamma) = \frac{\partial\gamma}{\partial w} A(\gamma) \ ,$$

which as a function of w is meromorphic on the twofold covering of the w-plane. Parametrize the local expansion of A_w around one of its singularities γ_j as:

$$A_w(\gamma) = \sum_{k=1}^{r_j} \frac{A_j^{(w)k}}{(w-w_j)^k} + \mathcal{O}((w-w_j)^0) \qquad \text{for} \quad \gamma \sim \gamma_j \ . \tag{6.38}$$

Then we find that the coefficients $A_j^{(w)k}$ of the local expansion of A_w have no explicit time dependence, i.e.

$$\partial_\pm A_j^{(w)k} \ = \ \left\{ A_j^{(w)k}, \mathcal{H}_\pm \right\} \ . \tag{6.39}$$

They satisfy the same Poisson structure as the A_j^k (6.37):

$$\left\{ (A_i^{(w)k})^A \, , \, (A_j^{(w)l})^B \right\} \ = \ \left\{ \begin{array}{ll} \delta_{ij} f^{AB}{}_C \, (A_j^{(w)k+l-1})^C & \text{for } k+l-1 \le r_j \\ 0 & \text{for } k+l-1 > r_j \end{array} \right. \ . \tag{6.40}$$

Thus, also in this case one there is a complete set of canonical explicitly time-independent variables.

The coset structure

To this point the isomonodromic ansatz has ignored the coset structure of the original model. The solution M of (2.22) which is obtained from the new basic object $A(\gamma)$ via (6.23) will in general not satisfy the original symmetry (2.43) which characterized the coset model. Thus, the new description still carries too many degrees of freedom. Here, we show how to cure this.

As functions of the original fields, the new variables $A(\gamma)$ have been defined only up to the freedom (3.4) in the original linear system so far. The entire structure described above remains invariant under this freedom. As it turns out [71], the restriction of this multiplicative freedom which is consistent with the isomonodromic truncation of this chapter is the condition (3.16) used in the approach of Belinskii and Zakharov. In terms of the isomonodromic objects, this condition reads

$$\Psi(\gamma)\, \tau\Big(\Psi^{-1}(\tfrac{1}{\gamma}) \Big) \ = \ M \ , \tag{6.41}$$

$$\gamma^2 A(\gamma) + M\, \tau\Big(A(\tfrac{1}{\gamma}) \Big)\, M^{-1} \ = \ 0 \ . \tag{6.42}$$

The second equation is obtained from derivation of the first. In particular, this last equation yields

$$\mathcal{V}^{-1} A(\gamma = \pm 1) \mathcal{V} \ \in \ \mathfrak{k} \ . \tag{6.43}$$

Recalling that $\partial_\pm M M^{-1} = 2\mathcal{V}P_\pm\mathcal{V}^{-1}$ we see that this condition is indeed suffient to guarantee that the matrix M obtained by integration satisfies the symmetry (2.43). The condition (6.42) takes a simpler form in terms of the variables $\widehat{A}(\gamma) \equiv \mathcal{V}A(\gamma)\mathcal{V}^{-1},$ where it reads

$$\gamma^2 \widehat{A}(\gamma) + \tau\Big(\widehat{A}(\tfrac{1}{\gamma})\Big) \;=\; 0\;. \tag{6.44}$$

Unfortunately the Poisson structure (6.27) is not automatically compatible with the condition (6.42). We may however treat the whole system as a constrained system, where (6.42) then builds a set of second-class constraints. Applying the canonical Dirac procedure [26] finally yields the following modified bracket on the phase space [74]

$$\begin{aligned}\Big\{ \overset{1}{\widehat{A}}(\gamma)\,,\, \overset{2}{\widehat{A}}(\mu)\Big\} \;=\;& \frac{1}{2}\Big[\,\frac{\Omega_{\mathfrak{g}}}{\gamma-\mu}\,,\, \overset{1}{\widehat{A}}(\gamma)+\overset{2}{\widehat{A}}(\mu)\,\Big] \\ &+\frac{1}{2}\Big[\,\Omega_{\mathfrak{g}}^{\tau}\,,\,\frac{\gamma}{1-\gamma\mu}\,\overset{1}{\widehat{A}}(\gamma)\;-\;\frac{\mu}{1-\gamma\mu}\,\overset{2}{\widehat{A}}(\mu)\Big]\end{aligned} \tag{6.45}$$

This structure indeed is compatible with (6.42). There remains the following set of first-class constraints (contained in (6.44) at $\gamma\to\infty$)

$$\widehat{A}_\infty + \tau(\widehat{A}_\infty) \;\equiv\; \lim_{\gamma\to\infty}\Big(\gamma\widehat{A}(\gamma) + \gamma\,\tau(\widehat{A}(\gamma))\Big) \;\approx\; 0\;, \tag{6.46}$$

which via (6.45) generate the $\mathbf{H}$-gauge transformations (2.65). This is the proper substitution of (6.25) after implementing the coset structure.

Thus, we have reduced the degrees of freedom so as to match the situation of the coset model.

6.3 Poisson algebra of observables

In the model as presented so far, observables can be defined in the sense of Dirac as objects that have vanishing Poisson bracket with all the constraints including the Hamiltonian constraints (6.34), which even play the most important role here. In two-time formalism this condition shows the observables to have no total dependence on $x^\pm$. This is a general feature of a covariant theory, where time dynamics is nothing but unfolding of a gauge transformation, and observables are the gauge invariant objects.

Regarding the connection $A(\gamma)$ as fundamental variables of the theory, the natural objects to build observables from are the monodromies of the linear system (6.21). They are given as

$$\Psi(\gamma)\mapsto\Psi(\gamma)M_\ell \qquad \text{for } \gamma \text{ running along the closed path } \ell\;. \tag{6.47}$$

Due to their definition these objects have no total $x^\pm$-dependence; in the isomonodromic sector which we treat here, the w-dependence is also absent.

For the simple pole sector let us denote by $M_i \equiv M_{\ell_i}$ the monodromies corresponding to the closed paths ℓ_i which respectively encircle the singularities γ_i and touch in one common basepoint. The remaining constraint of the theory which should have vanishing Poisson bracket with the observables is the generator of the constant gauge transformations (6.25), under which the monodromies transform by a common constant conjugation. Thus the set of Wilson loops

$$\Big\{ \mathrm{tr} \prod_k M_{i_k} \, \Big| \, k, (i_1, \ldots, i_k) \Big\} \tag{6.48}$$

builds the set of observables for this sector of the theory.

The Dirac brackets (6.45) define a Poisson structure on the monodromy matrices M_j. Rather then directly computing this bracket, we alternatively first obtain the Poisson structure on the monodromy matrices which is implied by (6.27). The Dirac bracket on the space of observables can then be deduced by simple symmetry arguments.

Let $A(\gamma)$ be a connection on the punctured plane $\gamma/\{\gamma_1, \ldots, \gamma_N\}$, equipped with the Poisson structure:

$$\Big\{ \overset{1}{A}(\gamma)\, , \overset{2}{A}(\mu) \Big\} \;=\; \Big[\, \frac{\Omega_{\mathfrak{g}}}{\gamma - \mu} \, , \, \overset{1}{A}(\gamma) + \overset{2}{A}(\mu) \Big] \, . \tag{6.49}$$

Let further Ψ be defined as solution of the linear system

$$\partial_\gamma \Psi(\gamma) = A(\gamma) \Psi(\gamma) \; , \tag{6.50}$$

normalized at a fixed basepoint s_0

$$\Psi(s_0) = I \; , \tag{6.51}$$

and denote by $M_1, \ldots, M_N$ the monodromy matrices of Ψ corresponding to a set of paths with endpoint s_0, which encircle $\gamma_1, \ldots, \gamma_N$, respectively. Ensure holomorphy of Ψ at ∞ by the first-class constraint

$$A_\infty = \lim_{\gamma \to \infty} \gamma A(\gamma) = 0 \; . \tag{6.52}$$

Then, in the limit $s_0 \to \infty$, the Poisson structure of the monodromy matrices is given by:

$$\Big\{ \overset{1}{M}_i \, , \, \overset{2}{M}_i \Big\} \;=\; \mathrm{i}\pi \Big(\overset{2}{M}_i \, \Omega_{\mathfrak{g}} \, \overset{1}{M}_i - \overset{1}{M}_i \, \Omega_{\mathfrak{g}} \, \overset{2}{M}_i \Big) \; , \tag{6.53}$$

$$\Big\{ \overset{1}{M}_i \, , \, \overset{2}{M}_j \Big\} \;=\; \mathrm{i}\pi \Big(\overset{1}{M}_i \, \Omega_{\mathfrak{g}} \, \overset{2}{M}_j + \overset{2}{M}_j \, \Omega_{\mathfrak{g}} \, \overset{1}{M}_i - \Omega_{\mathfrak{g}} \, \overset{1}{M}_i \overset{2}{M}_j - \overset{1}{M}_i \overset{2}{M}_j \, \Omega_{\mathfrak{g}} \Big) \quad \textit{for} \;\; i < j \; , \tag{6.54}$$

where the paths defining the monodromy matrices M_i are ordered with increasing i with respect to the distinguished path $[s_0 \to \infty]$.

Here, we collect several comments on this result, whereas for the proof we refer to [74].

Remark 6.5 The first-class constraint (6.52) generates constant gauge transformations of the connection A in the Poisson structure (6.49). In terms of the monodromy matrices, holomorphy of Ψ at ∞ is reflected by

$$M_\infty \equiv \prod M_i = I \; , \tag{6.55}$$

which in turn is a first-class constraint and generates the action of constant gauge transformations on the monodromy matrices in the structure (6.53) and (6.54). The ordering of this product is fixed to coincide with the ordering that defines (6.54). In accordance with (6.48), the structure (6.53), (6.54) implies

$$\Big\{ M_\infty \,,\, \mathrm{tr} \prod_k M_{i_k} \Big\} = 0 \;. \tag{6.56}$$

Remark 6.6 The evident asymmetry of (6.54) with respect to the interchange of i and j is due to the fact, that the monodromy matrices are defined by the homotopy class of the path, which connects the encircling path with the basepoint in the punctured plane. This gives rise to a cyclic ordering of the monodromies.

The distinguished path $[s_0 \to \infty]$ breaks and thereby fixes this ordering. It is remnant of the so-called eyelash that enters the definition of the analogous Poisson structure in the combinatorial approach [43, 2], being attached to every vertex and representing some freedom in this definition. However, the choice of another path $[s_0 \to \infty]$ simply corresponds to a global conjugation by some product of monodromy matrices: a shift of this eyelash by j steps corresponds to the transformation

$$M_k \to (M_1 \dots M_j)^{-1} M_k (M_1 \dots M_j) \;.$$

Therefore the restricted Poisson structure on gauge invariant objects is independent of this path.

Remark 6.7 A seeming obstacle of the structure (6.53), (6.54) is the violation of Jacobi identities. Actually, this results from heavily exploiting the constraint (6.52) in the calculation of the Poisson brackets. As therefore these brackets are valid only on the first-class constraint surface (6.55), Jacobi identities can not be expected to hold in general.

However, the same reasoning shows, that the structure (6.53), (6.54) restricts to a Poisson structure fulfilling Jacobi identities on the space of gauge invariant objects. On this space, the structure reduces to the original Goldman bracket [48] and coincides with the restrictions of previously found and studied structures on the monodromy matrices [43]:

$$\begin{aligned} \Big\{ \overset{1}{M}_i \,,\, \overset{2}{M}_i \Big\} &= \overset{2}{M}_i\, r_+ \overset{1}{M}_i + \overset{1}{M}_i\, r_- \overset{2}{M}_i - r_- \overset{1}{M}_i \overset{2}{M}_i - \overset{1}{M}_i \overset{2}{M}_i\, r_+ \\ \Big\{ \overset{1}{M}_i \,,\, \overset{2}{M}_j \Big\} &= \overset{1}{M}_i\, r_+ \overset{2}{M}_j + \overset{2}{M}_j\, r_+ \overset{1}{M}_i - r_+ \overset{1}{M}_i \overset{2}{M}_j - \overset{1}{M}_i \overset{2}{M}_j\, r_+ \\ &\quad \text{for } \; i<j \;, \end{aligned} \tag{6.57}$$

where r_+ and $r_- \equiv -\Pi r_+ \Pi$ are arbitrary solutions of the classical Yang-Baxter equation

$$[r_{12}, r_{23}] + [r_{12}, r_{13}] + [r_{13}, r_{23}] = 0. \tag{6.58}$$

and the symmetric part of r_+ is required to be $\mathrm{i}\pi\Omega_{\mathfrak{g}}$. With $r_+ \equiv \mathrm{i}\pi\Omega_{\mathfrak{g}}$, (6.57) reduces to (6.53), (6.54) such that our structure is in some sense the skeleton, which may be dressed with additional freedom that vanishes on gauge invariant objects. On the space of monodromy matrices themselves, introduction of r-matrices may be considered as some regularization to restore associativity, whereas the fact that $\Omega_{\mathfrak{g}}$ itself does not satisfy the classical Yang-Baxter equation is equivalent to (6.53), (6.54) not obeying Jacobi identities.

Remark 6.8 For eventually treating the coset model, the following additional structure is important. There is an involution $\tilde{\tau}$ on the set of observables, defined by the cyclic shift $M_i \mapsto M_{i\pm n}$, where $N = 2n$ is the total number of monodromies. This involution is an automorphism of the Poisson structure on the algebra of observables:

$$\{\tilde{\tau}(X_1), \tilde{\tau}(X_2)\} = \tilde{\tau}\left(\{X_1, X_2\}\right) \ , \tag{6.59}$$

for X_1, X_2 being traces of arbitrary products of monodromy matrices. This is a corollary of Remark 6.6, as it follows from the invariance of the Poisson structure on gauge invariant objects with respect to a shift of the eyelash that defines the ordering of monodromy matrices. Like every involution, $\tilde{\tau}$ defines a grading of the algebra into its eigenspaces of eigenvalue ± 1. In particular, the even part forms a closed subalgebra.

The final goal of this section is the computation of the Dirac bracket on the space of monodromy matrices. Let us first state the implications of the coset structure on this space. In the sector of simple poles, (6.41) implies that the singularities appear in pairs with

$$\gamma_j = \frac{1}{\gamma_{j+n}} \ , \tag{6.60}$$

(where $N = 2n$ is the number of singularities), while the corresponding monodromies are related by

$$M_{j+n} = \tau(M_j) \ . \tag{6.61}$$

To apply the result (6.53), (6.54) the corresponding paths must be chosen pairwise symmetric under $\gamma \mapsto \frac{1}{\gamma}$. This uniquely relates the ordering of the monodromy matrices in (6.54) to the ordering defined by (6.60).

The Dirac bracket now follows from simple symmetry arguments avoiding the direct computation for objects that are invariant under $\mathbf{G}$-valued gauge transformations (i.e. traces of arbitrary products of M_j). The involution τ^∞ introduced by (3.8) acts on M_j according to (6.41) as follows:

$$\tau^\infty(M_j) = \tau(M_{j+n}) \ . \tag{6.62}$$

Therefore, the set of all $\mathbf{G}$-invariant functionals of M_j may be represented as

$$M_{\text{S}} \oplus M_{\text{AS}} \ , \tag{6.63}$$

with eigenvalues ± 1 under τ^∞, respectively. Since τ is an automorphism of the structure (6.53), (6.54), the definition of τ^∞ in (6.62) implies (taking into account Remark 6.8)

$$\{M_{\text{S}}, M_{\text{S}}\} \subseteq M_{\text{S}} \ , \qquad \{M_{\text{S}}, M_{\text{AS}}\} \subseteq M_{\text{AS}} \ , \qquad \{M_{\text{AS}}, M_{\text{AS}}\} \subseteq M_{\text{S}} \ . \tag{6.64}$$

The constraints (6.61) are equivalent to vanishing of M_{AS}; therefore the part of $\mathbf{G}$-invariant variables surviving after the Dirac procedure is contained in M_{S}. The former Poisson bracket (6.53), (6.54) on M_{S} coincides with the Dirac bracket.

6.4 Quantization

In this section we describe different quatization procedures for the isomonodromic sector of the model with simple poles. For simplicity and illustration we first recall the canonical quantization of the Poisson brackets (6.32), where the coset structure (6.42) is ignored for a while [71]. Like the quantization of (6.13) this yields a link to the Knizhnik-Zamoldchikov system. We continue with identifying the quantum analogues of the monodromy matrices in this representation and work out their algebraic structure. This may be compared with a direct quantization of the monodromy algebra (6.53), (6.54) or (6.57), respectively. Finally, we give the necessary modifications to properly include the coset structure of the model (6.42).

Quantum connection

We briefly describe the quantization of the model in the isomonodromic sector with only simple poles [71]. Straightforward quantization of the linear Poisson brackets (6.32) leads to the following commutation relations:

$$\left[A_i^a\,, A_j^b\,\right] \;=\; \mathrm{i}\hbar\delta_{ij}f^{abc}A_j\;, \qquad [\rho^\pm, \partial_\pm\sigma] \;=\; -\mathrm{i}\hbar\;. \tag{6.65}$$

Accordingly we represent the $\rho^\pm$ by multiplication operators, and further define

$$A_j^A \equiv \mathrm{i}\hbar t_j^A\;, \qquad \partial_\pm\sigma \equiv \mathrm{i}\hbar\frac{\partial}{\partial\rho^\pm}\;, \tag{6.66}$$

where t_j^A acts on a representation V_j of the algebra $\mathfrak{g}$. Thus, the quantum state $\psi(\rho^\pm)$ in a sector with given singularities depends on the fields $\rho^\pm$ and lives in the tensor-product

$$V^{(N)} \equiv V_1 \otimes \ldots \otimes V_N\;, \tag{6.67}$$

of N representation spaces.

The whole "dynamics" of the theory is now encoded in the constraints (6.34), which accordingly play the role of the Wheeler-DeWitt equations here:

$$\mathcal{C}_\pm\psi = 0\;, \tag{6.68}$$

and can be written out in explicit form using (6.34), (6.29), (6.66):

$$\frac{\partial}{\partial\rho^\pm}\,\psi(\rho^\pm) \;=\; 2\mathrm{i}\hbar\,\rho^{-1}\sum_{k\neq j}\frac{\Omega_{jk}}{(1\pm\gamma_j)(1\pm\gamma_k)}\,\psi(\rho^\pm)\;, \tag{6.69}$$

where Ω_{jk} is defined as in (6.17).

The other constraint that restricts the physical states arrives from (6.25); in the quantized sector it is reflected by:

$$\Big(\sum_j t_j^A\Big)\;\psi(\rho^\pm) = 0\;. \tag{6.70}$$

The general solution of the system (6.69) is not known. However, these equations turn out to be intimately related to the Knizhnik-Zamolodchikov system (6.17). Namely, if φ_{KnZ}

is a $V^{(N)}$-valued function of $\gamma_1, \dots, \gamma_N$, which solves (6.17) and the constraint (6.70), and if further the γ_j depend on $x^\pm$ according to (3.3), then

$$\psi = \prod_{j=1}^{N} \Big(\frac{\partial \gamma_j}{\partial w_j}\Big)^{\frac{1}{2}\mathrm{i}\hbar\Omega_{jj}} \varphi_{\mathrm{KnZ}} \;, \tag{6.71}$$

solves the constraint equations (6.69) [71]. The Casimir operator Ω_{jj} defined above is assumed to act diagonal on the states, for $\mathfrak{g}$=$\mathfrak{sl}(2)$ for example, this is simply $\Omega_{jj} = \frac{1}{2}s_j(s_j{-}2)$, classifying the representation.

Quantum monodromy matrices

Having quantized the connection $A(\gamma)$ as described in the previous section, it is a priori not clear how to identify quantum operators corresponding to the classical monodromy matrices in this picture. As they are classically highly nonlinear functions of the A_j, arbitrarily complicated normal-ordering ambiguities may arise in the quantum case.

We choose a simple convention, replacing the classical linear system

$$\partial_\gamma \Psi(\gamma) = A(\gamma)\Psi(\gamma) \;, \tag{6.72}$$

by formally the same one, where all the arising matrix entries are operators now, i.e. (6.72) is an operator on $V_0 \otimes V^{(N)}$ where V_0 denotes the (classical) vector space, already necessary for the definition of (6.10), (6.20) and the (quantum) part $V^{(N)}$ has been defined in(6.67).

We have thereby fixed the operator ordering on the right hand side in what seems to be a rather natural way. In the same way, we define the quantum monodromy matrices to be given by

$$\Psi(\gamma) \mapsto \Psi(\gamma)M_j \;, \quad \text{for } \gamma \text{ encircling } \gamma_j \;, \tag{6.73}$$

where the (quantum) Ψ-function is normalized as

$$\Psi(\gamma) = \Big(I + \mathcal{O}(\tfrac{1}{\gamma})\Big)\, \gamma^{-A_\infty} \qquad \text{around } \gamma \sim \infty \;. \tag{6.74}$$

Remark 6.9 The normalization condition (6.74) generalizes the one we chose in the classical case (6.51) where the basepoint s_0 was sent to infinity. This generalization is necessary, because the constraint (6.52) is not fulfilled as an operator identity in the quantum case, which means, that the quantum Ψ-function as an operator is definitely singular at $\gamma = \infty$ with the behavior (6.74). Only its action on physical states, which are by definition annihilated by the constraint (6.25) may be put equal to the identity for $\gamma\!=\!\infty$.

We are interested in the algebraic structure of the quantum monodromy matrices M_j defined by (6.73). This follows from the observation [109] that the quantum linear system (6.72) is related to the Knizhik-Zamolodchikov systems with N and $N{+}1$ insertions, respectively, by

$$\Psi(\gamma, \gamma_1, \dots, \gamma_N) = \Big((I \otimes U_N^{-1}(\gamma_1, \dots, \gamma_N)\Big) U_{N+1}(\gamma, \gamma_1, \dots, \gamma_N) \;, \tag{6.75}$$

with the evolution operators U_N from (6.18). Having Remark 6.1 in mind, the quantum linear system may thus be understood as a mixture of the Schlesinger (6.13) and the Knizhnik-Zamolodchikov (6.17) system, where the former corresponds to the classical vector space V_0 with associated insertion γ and the latter corresponds to the quantum space (6.67).

In particular, (6.75) shows that the monodromies of (6.73) may be identified among the monodromies of the Knizhnik-Zamolodchikov system with $N+1$ insertions. It has been shown by Drinfeld, that these monodromies in turn are related to the braid group representations induced by certain quasi-bialgebras [30, 29].

Putting all these things together [74] we obtain the following algebraic structure

$$\begin{aligned} R_- \overset{1}{M}_i R_-^{-1} \overset{2}{M}_i &= \overset{2}{M}_i R_+ \overset{1}{M}_i R_+^{-1} , \\ R_+ \overset{1}{M}_i R_+^{-1} \overset{2}{M}_j &= \overset{2}{M}_j R_+ \overset{1}{M}_i R_+^{-1} , \qquad \text{for } i<j , \end{aligned} \tag{6.76}$$

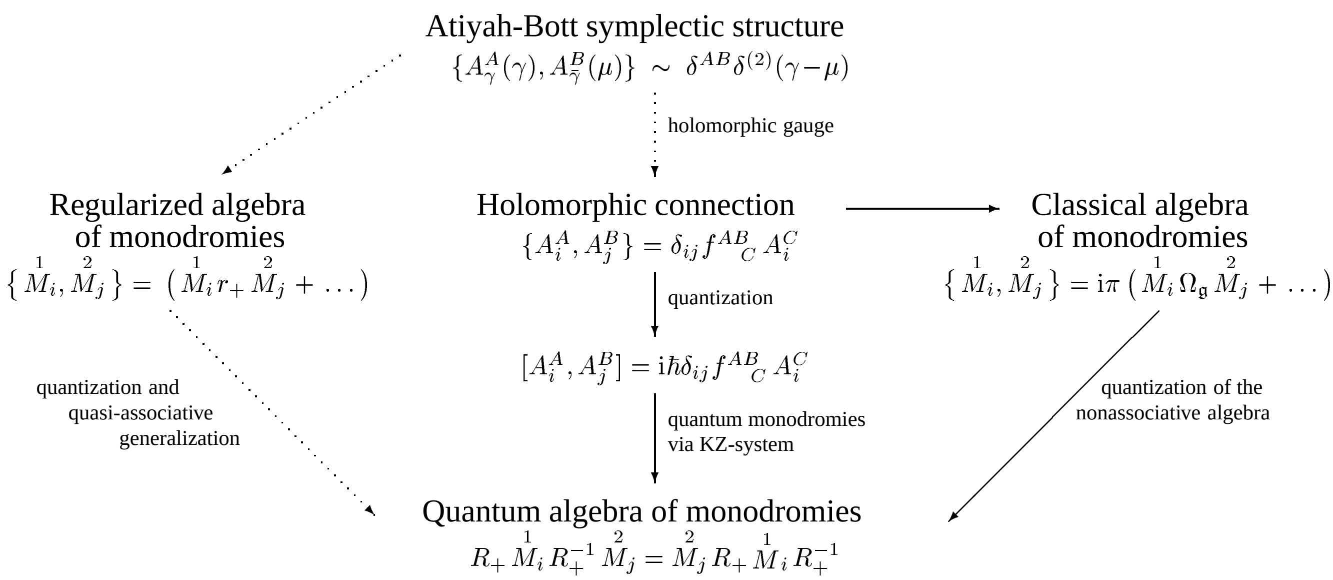
with the R-matrices $R_\pm$ given by

$$R_- \equiv \overset{2}{u} R_{\mathcal{U}}^{-1} \overset{1}{u}^{-1}, \qquad R_+ \equiv \Pi R_-^{-1} \Pi , \tag{6.77}$$

where $R_{\mathcal{U}}$ is the universal R-matrix of the so-called Drinfeld-Jimbo quantum enveloping algebra associated with $\mathfrak{g}$ [27, 57] and u is some automorphism on $V_0 \otimes V^{(N)}$. The classical limit of these R-matrices may be computed and yields

$$R_\pm = I \otimes I \pm (\mathrm{i}\hbar)(\mathrm{i}\pi\Omega_{\mathfrak{g}}) + \mathcal{O}_\pm(\hbar^2) . \tag{6.78}$$

Thus, we have obtained the quantum algebra of the quantum monodromy matrices by identifying the corresponding operators inside the picture of the quantized holomorphic connection $A(\gamma)$. The classical limit of this algebra coincides with the classical algebra of monodromy matrices (6.53), (6.54). This shows the "commutativity" of the (classical and quantum) links between the connection and the monodromies with the corresponding quantization procedures. Let us sketch this in the following diagram:



The dotted lines in this diagram depict the link to the usual way, quantum monodromies have been treated. As was sketched in Remark 6.7., their classical algebra can be derived from the original symplectic structure of the connection up to certain degrees of gauge freedom: for later restriction on gauge invariant objects, this algebra may be described with an arbitrary classical r-matrix. A direct quantization of this structure is provided by a structure of the form (6.76), where the quantum R-matrices live in the classical spaces only and admit the classical expansion $R_\pm = I + \mathrm{i}\hbar r_\pm + \mathcal{O}_\pm(\hbar^2)$ [1, 2].

In contrast to this quantum algebra which underlies (6.57), the R-matrices in (6.76) – due to the automorphism u – also act nontrivially on the quantum representation space. Their classical matrix entries may be considered as operator-valued, meaning, that the quantum algebra can be understood alternatively as nonassociative or as "soft". This is in some sense the quantum reason for the fact, that the classical algebra (6.53), (6.54) fails to satisfy Jacobi identities. However, note that (6.76) only describes the R-matrix in any fixed representation of the monodromies; for a description of the abstract algebra, compare the quasi-associative generalization in [2].

Quantum coset model

We have seen that the proper Poisson structure to be quantized for the coset model is (6.45). This goes along the same line as the quantization of (6.27) described above.

Having solved the constraints (6.42), the number of degrees of freedom is effectively reduced. The simple poles appear in pairs related by (6.60). Half of the residues of (6.30) is represented according to (6.65), while the other half is obtained via

$$\widehat{A}_j \;=\; \tau\Big(\widehat{A}_{j+n}\Big) \;. \tag{6.79}$$

The constraint equations (6.68) (the Wheeler-DeWitt equations here) take the form

$$\frac{\partial}{\partial\rho^\pm}\,\psi(\rho^\pm) \;=\; 2\mathrm{i}\hbar\rho^{-1}\left\{\sum_{j,k}\frac{(1+\gamma_j\gamma_k)\;\Omega_{jk}}{(1\pm\gamma_j)(1\pm\gamma_k)} - \sum_{j,k}\frac{(\gamma_j+\gamma_k)\;\Omega^\tau_{jk}}{(1\pm\gamma_j)(1\pm\gamma_k)}\right\}\psi(\rho^\pm)\,, \tag{6.80}$$

with Ω^τ from (3.47). Additionally, the physical states have to be annihilated by the first-class constraint (6.46):

$$\left(\sum_j t^A_j + \sum_j \tau\big(t^A_j\big)\right)\psi(\rho^\pm) \;=\; 0\;. \tag{6.81}$$

Modifying (6.71) we can establish a link between solutions of the quantum constraint equations (6.80), (6.81) (i.e. physical states) and solutions of what we will refer to as the Coset-Knizhnik-Zamolodchikov (CKZ) system [74]:

$$\frac{\partial\varphi_{\rm CKZ}}{\partial\gamma_j} = \mathrm{i}\hbar\left\{\sum_{k\neq j}\frac{1+\gamma_k/\gamma_j}{\gamma_j-\gamma_k}\,\Omega_{jk} + \sum_k\frac{\gamma_k+1/\gamma_j}{\gamma_j\gamma_k-1}\,\Omega^\tau_{jk}\right\}\;\varphi_{\rm CKZ}\;. \tag{6.82}$$

The precise relation to (6.80) is the following:

If φ_{CKZ} is a solution of (6.82) obeying the constraint (6.81), and the γ_j depend on $\rho^\pm$ according to (3.3), then

$$\psi = \prod_{j=1}^{n} \left(\gamma_j^{-1} \frac{\partial \gamma_j}{\partial w_j} \right)^{\mathrm{i}\hbar \Omega_{jj}} \varphi_{\text{CKZ}} \, , \tag{6.83}$$

solves the constraint (Wheeler-DeWitt) equations (6.80).

The procedure of identifying observables may be outlined just as in the case of the principal model. Again the monodromies of the quantum linear system are the natural candidates for building observables and contain a complete set for the simple pole sector. The actual observables are generated from combinations of matrix entries of these monodromies that commute with the constraint (6.81). From general reasoning according to the classical procedure, relevant objects turn out to be the combinations of $\mathbf{G}$-invariant objects, that are also invariant under the involution τ^∞.

6.5 Isomonodromic deformations and KZB equations on the torus

This section is based on [73]. We leave the concrete model of dimensionally reduced gravity and like in section 6.1 study abstract isomonodromic deformations. The scheme presented above allows natural extension to Riemann surfaces of genus one. Instead of the Knizhnik-Zamolodchikov system (6.17) on the sphere, in this case we obtain the link to the Knizhnik-Zamolodchikov-Bernard (KZB) system that has appeared in the study of the corresponding higher genus conformal field theories [8, 9]. The conceptual novelty of twisted functions, that is introduced in WZW conformal field theories on the torus in order to get a proper description of the action of inserted affine zero modes in the correlation functions, enters the game in a very natural way here.

In the context of dimensionally reduced gravity these structures may prove to be important in an isomonodromic approach to two-dimensional world-sheets with nontrivial topology. This extension would be indispensable for a "stringy" interpretation of the model.

Holomorphic gauge fixing

We start again from a smooth $\mathfrak{g}$-valued one-form A on the torus. To simplify notation and without loss of generality we restrict to the case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. In the explicit formulae we will use standard Chevalley generators $t^3, t^\pm$. Denote the periods of the torus by 1 and τ.

Holomorphic gauge $A_{\bar\gamma} = 0$ can not be achieved in general. However, taking into account our remarks from the previous section, the essential fact is,[41] that a dense subspace of smooth (0,1)-forms can be gauged into constants of the form

$$A_{\bar\gamma} = \frac{2\pi \mathrm{i}\lambda}{\tau - \bar\tau}\, \sigma_3 \, , \qquad \lambda \in \mathbb{C} \, . \tag{6.84}$$

The holomorphic gauge condition would require an additional gauge transformation of the kind $g = \exp(2\pi\mathrm{i}\lambda \frac{\gamma - \bar\gamma}{\tau - \bar\tau} \sigma_3)$. This is obviously multi-valued on the torus, having a multiplicative twist: $g \mapsto \exp(2\pi\mathrm{i}\lambda\sigma_3) g$ for γ encircling the fundamental $(0,\tau)$-cycle. The result of a gauge transformation of this kind is a twist in the remaining holomorphic (1,0)-form $A(\gamma)$:

$$A(\gamma+1) = A(\gamma) \qquad A(\gamma+\tau) = e^{2\pi\mathrm{i}\lambda\,\mathrm{ad}\sigma_3} A(\gamma)\ . \tag{6.85}$$

In components this reads:

$$A^3(\gamma+\tau) = A^3(\gamma) \qquad A^{\pm}(\gamma+\tau) = e^{\pm 4\pi\mathrm{i}\lambda} A^{\pm}(\gamma)\ .$$

Even though in principle gauge transformations must be defined globally single-valued in order to conserve physics, in this case the proceeding is justified by the fact, that the non-gauge-trivial part of $A_{\bar{\gamma}}$ survives as an arising twist of the holomorphic connection A. This is how the holomorphic gauge causes the appearance of twisted quantities in a rather natural way.

Some meromorphic functions on the torus

Before we start to investigate isomonodromic quantization on the torus, let us collect some simple facts about twisted meromorphic functions on the torus. A basic ingredient to describe functions of this kind, is Jacobi's theta-function:

$$\theta(\gamma) \equiv \sum_{n\in\mathbb{Z}} e^{2\pi\mathrm{i}(\frac{1}{2}n^2\tau + n\gamma)}\ ,$$

which is holomorphic, twisted as: $\theta(\gamma+1) = \theta(\gamma)$, $\theta(\gamma+\tau) = e^{-\mathrm{i}\pi(\tau+2\gamma)}\theta(\gamma)$ and has simple zeros for $\gamma \in \frac{1}{2}(\tau+1) + \mathbb{Z} + \tau\mathbb{Z}$.

Standard combinations are the functions [42]

$$\rho(\gamma) \equiv \frac{\theta'(\gamma - \frac{1}{2}(\tau+1))}{\theta(\gamma - \frac{1}{2}(\tau+1))} + \mathrm{i}\pi\ , \quad \text{and} \quad \sigma_\lambda(\gamma) \equiv \frac{\theta(\lambda - \gamma - \frac{1}{2}(\tau+1))\theta'(\frac{1}{2}(\tau+1))}{\theta(\gamma + \frac{1}{2}(\tau+1))\theta(\lambda - \frac{1}{2}(\tau+1))}\ ,$$

which have simple poles with normalized residue in $\gamma = 0$ and additive and multiplicative twist, respectively:

$$\rho(\gamma+\tau) = \rho(\gamma) - 2\pi\mathrm{i}\ , \qquad \sigma_\lambda(\gamma+\tau) = e^{2\pi\mathrm{i}\lambda}\sigma_\lambda(\gamma)\ .$$

Moreover, they satisfy

$$\rho(-\gamma) = -\rho(\gamma)\ , \qquad \sigma_\lambda(\gamma) = -\sigma_{-\lambda}(-\gamma)\ , \tag{6.86}$$

and the identity

$$\partial_\lambda\sigma_\lambda(x-y) = \sigma_\lambda(x-y)\Big(\rho(\gamma-x) - \rho(\gamma-y)\Big) - \sigma_{-\lambda}(\gamma-x)\sigma_\lambda(\gamma-y)\ . \tag{6.87}$$

These relations can be proved checking residues and twist properties. All the following calculations rely on the fact, that meromorphic functions on the torus with simple poles are uniquely determined by their residues if they are multiplicatively twisted, whereas functions with additive or vanishing twist are determined only up to constants. In generic situation there are no holomorphic twisted functions on the torus.

Isomonodromic deformations

Equipped with these tools we can now start to describe the twisted meromorphic connection $A(\gamma)$. Because of its twist properties (6.85), $A(\gamma)$ is of the form:

$$A^{\pm}(\gamma) \;=\; \sum_i A_i^{\pm}\sigma_{\pm 2\lambda}(\gamma-\gamma_i)\;, \quad A^3(\gamma) \;=\; \sum_i A_i^3\rho(\gamma-\gamma_i) - B^3\;. \tag{6.88}$$

Define again Ψ by the linear system

$$\partial\Psi(\gamma) = A(\gamma)\Psi(\gamma)\;. \tag{6.89}$$

The function Ψ will get monodromies M_i and $M_{(0,1)}$ from the right hand side, if γ encircles γ_i or the $(0,1)$ cycle of the torus. If γ runs along the $(0,\tau)$ cycle, Ψ will exhibit an additional left monodromy due to the twist (6.85) of A:

$$\Psi(\gamma) \mapsto e^{2\pi \mathrm{i}\lambda\sigma_3}\Psi(\gamma)M_{(0,\tau)}\;. \tag{6.90}$$

Under isomonodromic deformation we will understand the invariance of the right hand side monodromy data under the change of the parameters of the punctured torus, which are the singular points γ_i and the period τ. The connection data in this case are the residues A_i, the additive constant B^3 and the twist λ.

Let us first investigate their γ_i-dependence. In addition to the residues of $\partial_i\Psi\Psi^{-1}$ we have to determine its twist around $(0,\tau)$ from isomonodromy conditions. Equation (6.90) yields:

$$\Big(\partial_i\Psi\Psi^{-1}\Big)(\gamma) \mapsto e^{2\pi\mathrm{i}\lambda\;\mathrm{ad}\sigma_3}\Big(\partial_i\Psi\Psi^{-1}\Big)(\gamma) + 2\pi\mathrm{i}\partial_i\lambda\sigma_3$$

This determines the form of the γ_i-dependence of Ψ to be:

$$\begin{aligned}\Big(\partial_i\Psi\Psi^{-1}\Big)^{\pm}(\gamma) &= -A_i^{\pm}\sigma_{\pm 2\lambda}(\gamma-\gamma_i)\;, \\ \Big(\partial_i\Psi\Psi^{-1}\Big)^{3}(\gamma) &= -A_i^3\rho(\gamma-\gamma_i) + B_i^3\;,\end{aligned} \tag{6.91}$$

and further on yields the γ_i-dependence of the twist parameter λ:

$$\partial_i\lambda = A_i^3\;. \tag{6.92}$$

We can now proceed as on the sphere in section 6.1. Compatibility of the equations (6.89) and (6.91) implies the following Schlesinger equations on the torus:

$$\begin{aligned}\partial_i A_j^3 &= -A_i^+A_j^-\sigma_{2\lambda}(\gamma_j-\gamma_i) + A_i^-A_j^+\sigma_{-2\lambda}(\gamma_j-\gamma_i)\;, \quad \text{for } j\neq i\;, \\ \partial_i A_i^3 &= \sum_{j\neq i} A_i^+A_j^-\sigma_{2\lambda}(\gamma_j-\gamma_i) - \sum_{j\neq i} A_i^-A_j^+\sigma_{-2\lambda}(\gamma_j-\gamma_i)\;, \\ \partial_i A_j^{\pm} &= \pm 2A_i^{\pm}A_j^3\sigma_{\pm 2\lambda}(\gamma_j-\gamma_i) \mp 2A_i^3A_j^{\pm}\rho(\gamma_j-\gamma_i) \pm 2B_i^3A_j^{\pm}\;, \quad \text{for } j\neq i\;, \\ \partial_i A_i^{\pm} &= \pm 2\sum_{j\neq i} A_i^{\pm}A_j^3\rho(\gamma_i-\gamma_j) \mp 2\sum_{j\neq i} A_i^3A_j^{\pm}\sigma_{\pm 2\lambda}(\gamma_i-\gamma_j) \\ &\quad \mp 2B^3A_i^{\pm} \pm 2B_i^3A_i^{\pm}\;, \\ \partial_i B^3 &= \frac{1}{2}\sum_{j\neq i}\Big(A_i^-A_j^+\partial_\lambda\sigma_{2\lambda}(\gamma_i-\gamma_j) - A_i^+A_j^-\partial_\lambda\sigma_{2\lambda}(\gamma_j-\gamma_i)\Big)\;,\end{aligned} \tag{6.93}$$

and a curvature condition on the constants B_i

$$\partial_i B_j^3 - \partial_j B_i^3 = \tfrac{1}{2} A_i^- A_j^+ \partial_\lambda \sigma_{2\lambda}(\gamma_i - \gamma_j) - \tfrac{1}{2} A_i^+ A_j^- \partial_\lambda \sigma_{2\lambda}(\gamma_j - \gamma_i) \ . \tag{6.94}$$

The equations (6.92), (6.93) and (6.94) build a system of differential equations that is automatically compatibel, just as the Schlesinger equations (6.13) on the sphere are. This may be directly checked by a rather lengthy but straightforward calculation, making repeated use of (6.86) and (6.87). Compatibility is valid on the constraint surface

$$\sum_j A_j^3 = 0 \ , \tag{6.95}$$

that was already implied by consistency of the twist properties of $A(\gamma)$ with the ansatz (6.88). This constraint here appears in a weaker form than on the sphere (6.2). This corresponds to the fact that the gauge freedom (6.7) has been fixed more rigorously on the torus in order to diagonalize the twist around the $(0, \tau)$-cycle. Here, the remaining constraint (6.95) generates those gauge transformations which are compatible with (6.84)

As on the sphere, it is possible to formulate the dependence (6.93) as a multi-time Hamiltonian structure. The Hamiltonians read

$$\begin{aligned} H_i \;=\; & \sum_{j \neq i} \Big(2 A_i^3 A_j^3 \rho(\gamma_i - \gamma_j) + A_i^+ A_j^- \sigma_{-2\lambda}(\gamma_i - \gamma_j) + A_i^- A_j^+ \sigma_{2\lambda}(\gamma_i - \gamma_j) \Big) \\ & - 2B^3 A_i^3 + 2B_i^3 \sum_j A_j^3 \ , \end{aligned} \tag{6.96}$$

and generate the γ_i-flows (6.93) in the Poisson structure

$$\{A_i^A, A_j^B\} \;=\; \delta_{ij} f^{AB}{}_C \, A_i^C \ , \qquad \{\lambda, B^3\} \;=\; \tfrac{1}{2} \ . \tag{6.97}$$

This structure arises from holomorphic gauge-fixing of the original bracket (6.6) in the same way, as does the bracket (6.3) on the sphere. In particular, remembering the origin of λ (6.84), the second equation may be viewed as a reminiscent of (6.6) for the constant modes of A_γ and $A_{\bar{\gamma}}$.

In analogy with (6.5) this Poisson structure admits a generalized r-matrix formulation

$$\Big\{ \overset{1}{A}(\gamma) \, , \, \overset{2}{A}(\mu) \Big\} \;=\; \Big[r(\gamma - \mu) \, , \, \overset{1}{A}(\gamma) + \overset{2}{A}(\mu) \Big] - \partial_\lambda r(\gamma - w) \Big(\sum_j A_j^3 \Big) \ , \tag{6.98}$$

with the twisted r-matrix

$$r(\gamma) = \tfrac{1}{2} \rho(\gamma) (t^3 \otimes t^3) + \sigma_{2\lambda}(\gamma)(t^+ \otimes t^-) + \sigma_{-2\lambda}(\gamma)(t^- \otimes t^+) \ . \tag{6.99}$$

In some sense this restricts to a classical r-matrix formulation on the constraint surface (6.95). Validity of the Jacobi identities is expressed by a twisted version of the classical Yang-Baxter equation.

The Hamiltonians (6.96) show the role of the constants B_i as parameters of gauge transformations generated by the first-class constraint (6.95). This suggests to simply skip these terms from the Hamiltonians, as is in fact done in the sequel, leading to the KZB equations. As a consequence, these truncated Hamiltonians only commute up to (6.95), meaning that the generated γ_i dynamics of the connection data produces isomonodromic deformation only up to certain shifts in the gauge orbit.

Finally we study isomonodromic deformation with respect to a change in the period τ of the torus. This can be done in complete analogy with the just treated case. From (6.90) the twist of $\partial_\tau \Psi\Psi^{-1}$ around $(0,\tau)$ turns out to be

$$\Big(\partial_\tau \Psi\Psi^{-1}\Big)(\gamma) \;\mapsto\; e^{2\pi \mathrm{i}\lambda\ \mathrm{ad}\sigma_3}\Big(\partial_\tau \Psi\Psi^{-1}\Big)(\gamma) - e^{2\pi \mathrm{i}\lambda\ \mathrm{ad}\sigma_3} A(\gamma) + 2\pi \mathrm{i}\partial_\tau\lambda\sigma_3 \ ,$$

which leads to the following τ-dependence of the function Ψ:

$$\begin{aligned} 2\pi\mathrm{i}\Big(\partial_\tau \Psi\Psi^{-1}\Big)^{\pm}(\gamma) &= \mp\frac{1}{2}\sum_j A_j^{\pm}\partial_\lambda\sigma_{\pm 2\lambda}(\gamma-\gamma_j) \ , \qquad (6.100)\\ 2\pi\mathrm{i}\Big(\partial_\tau \Psi\Psi^{-1}\Big)^{3}(\gamma) &= \frac{1}{2}\sum_j A_j^{3}\Big(\rho(\gamma-\gamma_j)^2 - \wp(\gamma-\gamma_j)\Big) + B_\tau^3 \ , \end{aligned}$$

and determines the τ-dependence of the twist parameter

$$\partial_\tau\lambda = -\tfrac{1}{2\pi\mathrm{i}}B^3 \ . \qquad (6.101)$$

Compatibility of (6.89) and (6.100) now yields additional Schlesinger-type equations:

$$\begin{aligned} 2\pi\mathrm{i}\partial_\tau A_i^3 &= -\frac{1}{2}\sum_j A_i^+ A_j^- \partial_\lambda\sigma_{-2\lambda}(\gamma_i-\gamma_j) - \frac{1}{2}\sum_j A_i^- A_j^+ \partial_\lambda\sigma_{2\lambda}(\gamma_i-\gamma_j) \ , \qquad (6.102)\\ 2\pi\mathrm{i}\partial_\tau A_i^{\pm} &= \pm\sum_j A_i^{\pm}A_j^3\Big(\rho(\gamma_i-\gamma_j)^2 - \wp(\gamma_i-\gamma_j)\Big)\\ &\quad + \sum_j A_i^3 A_j^{\pm}\partial_\lambda\sigma_{\pm 2\lambda}(\gamma_i-\gamma_j) \pm 2B_\tau^3 A_i^{\pm} \ ,\\ 2\pi\mathrm{i}\partial_\tau B^3 &= -\frac{1}{8}\sum_{i,j}\Big(A_i^+A_j^-\partial_\lambda^2\sigma_{-2\lambda}(\gamma_i-\gamma_j) - A_i^-A_j^+\partial_\lambda^2\sigma_{2\lambda}(\gamma_i-\gamma_j)\Big) \ , \end{aligned}$$

together with a curvature condition for $\frac{1}{2\pi\mathrm{i}}\partial_i B_\tau^3 - \partial_\tau B_i^3$. Again, compatibility of the whole system of differential equations may be shown by a straightforward calculation.

With the Poisson structure already given in (6.97) this flow is generated by the Hamiltonian

$$\begin{aligned} 2\pi\mathrm{i}\, H_\tau &= \frac{1}{4}\sum_{i\neq j}\Big(A_i^+A_j^-\partial_\lambda\sigma_{-2\lambda}(\gamma_i-\gamma_j) - A_i^-A_j^+\partial_\lambda\sigma_{2\lambda}(\gamma_i-\gamma_j)\Big) \qquad (6.103)\\ &\quad + \frac{1}{2}\sum_{i,j}A_i^3A_j^3\Big(\rho(\gamma_i-\gamma_j)^2 - \wp(\gamma_i-\gamma_j)\Big) + B^3B^3 + 2B_\tau^3\sum_j A_j^3 \ , \end{aligned}$$

where again we will skip B_τ^3 under the above remarks.

Quantization and Knizhnik-Zamolodchikov-Bernard system

The canonical quantization of the described Hamiltonian structure now directly leads to the KZB-system, as we shall finally show. Quantization is again performed straightforwardly with (6.97) being replaced by

$$\left[A_i^A, A_j^B\right] \;=\; \mathrm{i}\hbar\delta_{ij} f^{AB}{}_C\, A_i^C\;, \qquad \left[\lambda, B^3\right] \;=\; \tfrac{1}{2}\mathrm{i}\hbar\;. \tag{6.104}$$

In the γ_i-independent Schrödinger representation of the operators they can be realized as

$$A_i^A \;=\; \mathrm{i}\hbar\, I\otimes\ldots\otimes t_i^A\otimes\ldots\otimes I\;, \qquad B^3 \;=\; -\tfrac{1}{2}\mathrm{i}\hbar\partial_\lambda\;, \tag{6.105}$$

acting on quantum states $|\omega\rangle$ that are λ-dependent sections of a $V^{(N)} \equiv \bigotimes_j V_j$ bundle over $X_1 \equiv \{\text{fundamental domain of } \tau\} \otimes \mathbb{C}^N \setminus \{\text{diagonal hyperplanes}\}$.

The quantization of (6.93) and (6.102) in the Schrödinger picture provides this bundle with the horizontal connection:

$$\begin{aligned} \partial_i|\omega\rangle \;=\; H_i\,|\omega\rangle \;&=\; \tfrac{1}{2}\mathrm{i}\hbar t_i^3\partial_\lambda\,|\omega\rangle + \mathrm{i}\hbar\sum_{j\neq i}\Theta^\gamma_{ij}(\gamma_i-\gamma_j,\tau,\lambda)\,|\omega\rangle\;, \\ 2\pi\mathrm{i}\partial_\tau|\omega\rangle \;=\; 2\pi\mathrm{i}H_\tau\,|\omega\rangle \;&=\; \tfrac{1}{4}\mathrm{i}\hbar\partial_\lambda^2\,|\omega\rangle + \mathrm{i}\hbar\sum_{i,j}\Theta^\tau_{ij}(\gamma_i-\gamma_j,\tau,\lambda)\,|\omega\rangle\;, \end{aligned} \tag{6.106}$$

with

$$\begin{aligned} \Theta^\gamma_{ij}(\gamma,\tau,\lambda) \;&=\; \tfrac{1}{2}\rho(\gamma)(t_i^3\otimes t_j^3) + \sigma_{-2\lambda}(\gamma)(t_i^+\otimes t_j^-) + \sigma_{2\lambda}(\gamma)(t_i^-\otimes t_j^+)\;, \\ \Theta^\tau_{ij}(\gamma,\tau,\lambda) \;&=\; \tfrac{1}{4}\partial_\lambda\sigma_{-2\lambda}(\gamma)(t_i^+\otimes t_j^-) - \tfrac{1}{4}\partial_\lambda\sigma_{2\lambda}(\gamma)(t_i^-\otimes t_j^+) \\ &\quad + \tfrac{1}{8}\left(\rho^2(\gamma)-\wp(\gamma)\right)(t_i^3\otimes t_j^3)\;, \end{aligned}$$

acting non-trivially on V_i and V_j.

This is the KZB connection, found in [8] as system of differential equations for character-valued correlation functions. The form (6.106) coincides exactly with the form presented in [42] for $\mathfrak{sl}(2,\mathbb{C})$. In particular, the term that includes the derivative with respect to the twist parameter λ is the explicit analogue of the action of affine zero modes on correlation functions in WZW models. We stress again that in contrast to the system (6.17) on the sphere these Hamiltonians only commute up to the constraint (6.95) which implies the fact that the KZB-connection is flat only as a connection on the subbundle of states annihilated by $\sum_j t_j^3$, see [42].

Let us close with the remark that this result suggests similar links between the quantization procedure of isomonodromic deformations on higher genus Riemann surfaces and the corresponding higher KZB equations [9]. See [55, 83, 117] for further work.

7 Conclusions and Outlook

Let us briefly summarize the main results obtained in this thesis.

- We have set up the canonical formalism for a general class of two-dimensional coset space σ-models coupled to dilaton-gravity, that arise from dimensional reduction of various gravity and supergravity theories. The canonical Poisson structure (2.57) and the gauge algebra of constraints (2.62)–(2.63) have served as the starting point for the entire treatment.

- A complete set of nonlocal integrals of motion has been identified classically among the transition matrices of the associated linear system. They have been shown to be invariant under the full gauge algebra of constraints (3.26). Moreover, in a rather direct and unusual way they encode physical information (3.59), which in spite of their spatially nonlocal origin (3.17) allows localization in the two-dimensional worldsheet.

- The classical Poisson algebra of these nonlocal charges is well-defined and in contrast to the related structures in the flat-space σ-models does not exhibit any ambiguities, in spite of similar non-ultralocal terms in the fundamental Poisson brackets (2.57). The coordinate dependence of the spectral parameter (3.3) plays an essential role for this regularity. The resulting algebra (3.60), (3.61) is related to the (semiclassical) Yangian double [27, 28].

- Since the nonlocal charges parametrize the phase space (at least in the sector which admits the particular gauge fixing (3.36)), the adjoint action of the algebra of charges on itself describes a transitive symmetry. The well-known action of the Geroch group is recovered as the associated Lie-Poisson action. This provides a canonical realization of the Geroch group, which is an indispensable tool for later quantization.

- We have shown that the entire structure allows generalization to the maximally supersymmetric extension of the model. The $N = 16$ superconformal constraint algebra has been worked out, and has been used to prove that the nonlocal charges – obtained in analogy to the bosonic case – are indeed supersymmetric. As a byproduct, this result has confirmed that the supersymmetric extension of the bosonic linear system (4.30) given in [98, 103] does not receive any quartic fermionic contributions but already captures the full supersymmetric theory. The Poisson algebra of charges has been shown to coincide with the one of the bosonic sector.

- Quantization of the classical structures has been achieved for the coset spaces $\mathbf{G}/\mathbf{H} = SL(N,\mathbb{R})/SO(N)$, resulting in a modified (twisted) version of the Yangian double with a particular value of the central extension (5.5), (5.6). The pivotal classical object – the monodromy matrix $\mathcal{M}_{\text{BM}}$ – has been recovered within the quantum algebra as a classical matrix with self-adjoint operator entries (5.9).

- The further program of classifying representations of the quantum algebra has been outlined for the simplest case $SL(2,\mathbb{R})$. Already on this level, one may recognize several features (in particular, the repeated occurrences of discrete nonlocal structures), which e.g. distinguish the model from the quantization of its linear (abelian) subsector. The latter has been under active investigation from the point of view of midi-superspace models of quantum gravity [79, 3].

- Within the isomonodromic approach initiated in [70, 71], we have analyzed the algebraic structure of observables on the classical and the quantum level. For quantization we have exploited the inherent link to a modified version of the Knizhnik-Zamolodchikov equations (6.82) making the underlying coset structure manifest. In the general framework of isomonodromic deformations, we have established a similar link to the Knizhnik-Zamolodchikov-Bernard equations on the torus. So far, we have not been able to embed these structures into the canonical framework.

There are many things which remain to be elaborated. An immediate aim is the study of the representation theory of the algebra (5.5)–(5.9) according to the program outlined in section 5.3. Certainly, the hope is that the requirement of unitarity with respect to the $*$-structure (5.10) will strongly restrict the choice of representations.

Within the appropriate representations, the next goal would be the construction of some analogue of coherent states. They should exhibit minimal quantum fluctuations around given classical solutions. The discussion of the symmetry structure in sections 3.4 and 5.3 suggests that the quantum counterpart of the Geroch group (5.40) will play a key role in generating these states, giving rise to a Hopf algebra generalization of the coherent states' concept. Obviously, the usual (linear) framework of coherent states is too narrow to cope with the quantization of Lie-Poisson symmetries. With coherent states at hand, one would finally be in position to study in detail how quantization affects the known classical solutions of gravity (at least under the above mentioned reservations).

For the maximally supersymmetric model described in Chapter 4 with the underlying coset space $\mathbf{G}/\mathbf{H} = E_{8(+8)}/SO(16)$, it remains to extend the quantization to higher-dimensional and, in particular, the exceptional Lie algebras. The quantization given in section 5.1 has been strongly supported by many well-known properties of the Yangian algebras associated with $SL(N,\mathbb{R})$. Unfortunately, less is known about the related structures for E_8; see however [18] for the construction of the associated R-matrix.

An interesting and somewhat complementary approach to the quantum model would involve the construction of the nonlocal charges in a quantum model based on the original physical currents (2.39), (2.57), rather then quantizing (3.60), (3.61) directly. In the sense of [85, 10], one would have to establish the nonlocal charges and their algebra after quantization and not before. Physical states would have to be identified in an "unphysical" Hilbert space as the kernel of the constraint algebra (2.62) and (4.27), respectively, while the quantum nonlocal conserved charges serve as a spectrum-generating algebra relating these states. So far, we have, in contrast, adopted a rather pragmatic point of view, by directly searching for the possible quantum algebras that may underlie the classical integrable structure, tacitly assuming that integrability survives quantization. This means e.g. that we have neglected any effects of potential anomalies that may obstruct integrability and the nonlocal symmetries in the quantum theory. It is at this stage, that the maximally supersymmetric extension described in Chapter 4 may play its full role (since on the level of conserved charges studied here, we have – somewhat surprisingly – not encountered any essential differences between the resulting structures in the supersymmetric model compared with the purely bosonic sector).

In view of potential higher-dimensional interpretations of these models [102], it would further be necessary to generalize the entire framework to arbitrary Riemann surfaces Σ playing the role of the two-dimensional world-sheet. So far, it is even unclear how to extend the setting to the (seemingly modest) modification of periodic boundary conditions. As we have discussed in section 3.2, in this class of models, periodicity of the physical fields does not imply periodicity of the connection of the linear system (3.1). The construction of conserved charges thus has to be modified in some rather nontrivial fashion. Since (3.59) has shown a link between the world-sheet and the spectral-parameter plane, one would expect the structures (3.60), (3.61) and (3.62) to be eventually replaced by a Poisson algebra, which should accordingly be compatible with some periodicity of the nonlocal charges in the spectral parameter plane.

Another highly interesting generalization would include the extension of the framework to those models which arise from a dimensional reduction that includes a timelike Killing vector field, i.e. which are formulated on a two-dimensional world-sheet Σ with Euclidean signature. At present, it seems rather subtle to rigorously establish a canonical framework in the sector of stationary solutions where the canonical time-dependence has been dropped by hand. On the other hand, it is certainly this sector which contains the most interesting physical solutions, in particular, the black holes.

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