Characterization of chaotic motions of a charged particle in Kerr black hole magnetosphere

Hiroko Koyama [1] and Masaaki Takahashi [2]

[1] *Department of Physics, Nagoya University, Nagoya 464–8602, Japan*
[2] *Department of Physics and Astronomy, Aichi University of Education, Kariya, Aichi 448–8542, Japan*

Abstract

We study the properties of chaos in the motions of a charged test particle confined in a dipole magnetic field around a Kerr black hole. We characterize the chaos using the power spectrum of the time series of the particle's position. We find that the pattern of the power spectrum shows not only white noise but also $1/f$ fluctuation, depending on the values of the system parameters (the black hole's spin, the strength of the magnetic field, the total energy, and the total angular momentum). So we succeed in classifying the chaotic motions into the two distinct types. One is $1/f$, and the other is white noise. Based on this classification, we obtain "phase diagram" for the properties of the chaos. This phase diagram enables us to guess the black hole's spin and the strength of the magnetic field by observing the dynamics of the charged particle, even if the motion is chaotic.

1 Introduction

Black hole and accretion disk system, like as a central engine of AGN, compact X-ray sources and GRB, is astrophysically important, and has been investigated by many authors. Observationally, we can obtain X-ray spectrum and time variability data, and near future we may see black hole shadow. We are interested in such magnetic phenomena near a black hole, and our motivation is to understand property of magnetosphere near a black hole. So we assume global magnetic fields in black-hole geometry. The problem is how to get peculiar informations about the black hole and the surrounding magnetic fields.

Now we go back to a basic subject that motions of a charged test particle in black hole magnetosphere. Firstly we consider test-particle motions around a Kerr black hole. In this system, number of spacetime dimension is 4, and number of constants of motion is also 4. That is, rest mass, energy, angular momentum, and Carter constant [1]. Then this system is integrable, and the particle's orbits are regular. Next, we consider charged-particle motions in the dipole magnetic field around a Kerr black hole [2]. In this system, number of spacetime dimension is 4, but number of constants of motion is 3. That is, rest mass, energy, and angular momentum. The separation of variable has not been found, and this system can show nonintegrability [3]. So the particle motions in this system can be chaotic and complicated. In this way, nature is filled with phenomena that exhibit chaotic behavior.

In roughly speaking, motions of a charged particle in dipole magnetic field near Kerr black hole can be explained as following [3]. A charged particle can be trapped in the doughnut-like shaped zones which is similar to Van Allen belt in Earh's magnetic field. The particle motions are combination of gyration, bouncing and drifting. The particle gyrates around the magnetic field line, oscillates in the poloidal plane along the magnetic field line, and drifts in the toroidal direction. Chaotic sea in the Poincare maps have been confirmed [4].

Having found the existence of chaotic motions, we should now characterize and quantify the chaos to clarify the effect of the black-hole spin and the magnetic field. Then, in this paper, we look for statistical laws in the chaotic motions in the dipole magnetic field around a Kerr black hole to classify the chaos. Indeed, we can hardly learn anything about the chaos if we judge it only from the randomness of the distribution of the points in Poincaré maps or the positiveness of the Lyapunov exponents. Not a few people believe that chaotic system is simply random and completely unpredictable. Of course, we cannot predict the time evolution of the state of the test particle exactly, when its system is chaotic. However,

[1] E-mail:hiroko@allegro.phys.nagoya-u.ac.jp
[2] E-mail:takahasi@phyas.aichi-edu.ac.jp

even in such cases, we can frequently find some statistical laws which are proper to the system. One possible measure of chaos is the power spectrum of the time series. In our previous paper [5], we have succeeded to classify the chaos in the motions of a spinning test particle around Schwarzschild black hole, using the power spectrum of the time series of the particle's position. We have found out that the pattern of the power spectra are divided into two distinct types depending on the system parameters (spin and angular momentum) [5]. One is $1/f$-type fluctuations and the other is white noise. In this paper, we apply this method to characterize chaos in the motions of a charged test particle confined in the dipole magnetic field around a Kerr black hole. Our goal is to clarify the effect of the spin of the black hole and the magnetic field into the chaos in the particle's motions.

Our strategy to characterize the chaos in this paper is as follows. First, we introduce the power spectrum of the time series of z components of the particle's position. Then we characterize the properties of the chaos in the charged-particle motions in the dipole magnetic field in Kerr spacetime, using the pattern of the power spectrum. It is found that the pattern of the power spectrum can be classified as $1/f$ or white noise. That is, we succeed in classifying the chaotic motions into two distinct types. Based on this classification, we plot phase diagrams for properties of the chaos.

2 Equations for a charged particle around a black hole

We solve the motion of a charged test particle in a dipole magnetic field around a Kerr black hole. The metric is written by the Boyer-Lindquist coordinates (t, r, θ, ϕ) with $c = G = 1$, and the non-zero components of the contravariant metric $g^{\mu\nu}$ are given by

$$g^{tt} = \frac{A}{\Delta\Sigma}, \quad g^{t\phi} = \frac{2Mar}{\Delta\Sigma}, \quad g^{\phi\phi} = -\frac{1-2Mr/\Sigma}{\Delta\sin^2\theta}, \quad g^{rr} = -\frac{\Delta}{\Sigma}, \quad g^{\theta\theta} = -\frac{1}{\Sigma}, \tag{1}$$

where $\Delta \equiv r^2 - 2Mr + a^2$, $\Sigma \equiv r^2 + a^2\cos^2\theta$, and $A \equiv (r^2+a^2)^2 - \Delta a^2 \sin^2\theta$. M is mass of the black hole, and a is the spin parameter. The Hamiltonian for the charged particle is

$$H = \frac{1}{2} g^{\mu\nu} (\pi_\mu - qA_\mu)(\pi_\nu - qA_\nu), \tag{2}$$

where π_μ is the canonical momentum, q is charge, and A_μ is the 4-potential of the electromagnetic field. The equations of motion are given by the Hamilton's equations,

$$\frac{dx^\mu}{d\lambda} = \frac{\partial H(x^\nu, \pi_\nu)}{\partial \pi_\nu}, \qquad \frac{d\pi^\mu}{d\lambda} = -\frac{\partial H(x^\nu, \pi_\nu)}{\partial x_\nu}. \tag{3}$$

The 4-momentum of a charged particle are given by

$$p^\mu \equiv \frac{dx^\mu}{d\lambda} = g^{\mu\nu}(\pi_\nu - qA_\nu). \tag{4}$$

The magnetic field configuration is assumed by dipole magnetic field [2], which is a solution of vacuum Maxwell equations in Kerr geometry. The non-zero components of A_μ are given by

$$A_t = \frac{-3\mu}{2\gamma^2\Sigma}\left\{ \left[r(r-M) + (a^2 - Mr)\cos^2\theta\right] \frac{1}{2\gamma} \ln\left(\frac{r-r_-}{r-r_+}\right) - (r - M\cos^2\theta)\right\}, \tag{5}$$

$$\begin{aligned} A_\phi = \frac{-3\mu\sin^2\theta}{4\gamma^2\Sigma}\Big\{&(r-M)a^2\cos^2\theta + r(r^2 + Mr + 2a^2) \\ &- \left[r(r^3 - 2Ma^2 + a^2 r) + \Delta a^2\cos^2\theta\right] \frac{1}{2\gamma} \ln\left(\frac{r-r_-}{r-r_+}\right)\Big\}, \end{aligned} \tag{6}$$

where μ is a dipole moment, $\gamma \equiv \sqrt{M^2 - a^2}$ and $r_\pm \equiv M \pm \gamma$.

The rest mass of the charged particle, m, is defined by $m^2 \equiv -p_\mu p^\mu$. m is constant. In addition, from the stationary and axial symmetry of both the electromagnetic field and the spacetime geometry, energy and angular momentum, $E \equiv \pi_t = p_t + qA_t$ and $L \equiv -\pi_\phi = -(p_\phi + qA_\phi)$, respectively, are also constants of motion. That is, number of constants of motion is 3. On the other hand, number of spacetime dimension is 4. Then, the particle's orbits in this system can be chaotic. We solve Eqs. (3) by the Runge-Kutta method numerically.

3 Phase diagram for the properties of chaos

In this section, we characterize the chaos in the charged particle motions in dipole magnetic field around Kerr black hole. Here we analyze the time series of the particle position. In order to that, first, we introduce the power spectrum. The power spectrum of the time series of z components of the particle's position, $P_z(\omega)$, is defined by

$$P_z(\omega) = \left| \frac{1}{T} \int_0^T z(t) e^{i\omega t} dt \right|^2 . \tag{7}$$

We test the pattern of the power spectrum $P_z(\omega)$ for various grid points in the parameter space, and show the results in Figs. 1 and 2. Here we define the parameter Q as $Q \equiv 3q\mu/(4M^2 m)$. The value of L/M is fixed to -7.

In Fig. 1 we test the pattern of the power spectrum $P_z(\omega)$ of the chaotic orbits for grid points in two-dimensional $(a/M, E/m)$ configuration. The value of Q is fixed to -30 in Fig. 1 . In the region where the symbols ($\bigcirc$) are marked the power spectrum $P_z(\omega)$ shows $1/f$-type spectrum. On the other hand, in the region where symbols ($\Box$) are marked, the power spectrum $P_z(\omega)$ shows white noise. At the points where the symbols ($\times$) are marked, the orbits are not chaotic but regular.

In Fig. 2 we test the pattern of the power spectrum $P_z(\omega)$ for grid points in two-dimensional $(a/M, Q)$ configuration. At the points where the symbols ($\bigcirc$) are marked, the $1/f$-type power spectrum is observed. At the points where the symbols (Δ) are marked, the $1/f$-type power spectra are observed for low energy, and the white-noise power spectra are observed for high energy. At the points where the symbols ($+$) are marked, the orbit apparently behaves regular for low energy, and the white-noise power spectra are observed for high energy.

Figs. 1 and 2 can be considered as "phase diagrams" for the properties of chaos. These phase diagrams illustrate the effect of the black-hole spin and the strength of the magnetic field. When the black hole is slowly rotating, or when the magnetic field is not weak, the pattern of the power spectrum $P_z(\omega)$ of the chaotic orbits shows $1/f$ fluctuation for low energy, and shows white noise for high energy. On the other hand, when the black hole is rapidly rotating and the magnetic field is weak, we cannot observe such $1/f$ fluctuations. The particle's orbits are regular for low energy, and $P_z(\omega)$ of the chaotic orbits shows white noise for high energy. These phase diagrams (Figs. 1, 2) enables us in principle to guess the black hole's spin and the strength of the magnetic field, even if the particle's motion is chaotic.

4 Summary

In this paper we have investigated the properties of chaos in the motions of a charged particle in dipole magnetic field around a Kerr black hole. We have characterized the chaos using the power spectrum of the time series of z components of the test particle's position, $P_z(\omega)$. We have found that the pattern of the power spectrum $P_z(\omega)$ can be divided into distinct 2 types, $1/f$ and white noise, depending on the system parameters (black hole's spin and magnetic field). Based on this classification, we have obtained "phase diagrams" for the property of chaos (Figs. 1, 2). These phase diagrams illustrate the effect of the black-hole spin and the strength of the magnetic field. The chaos we found in this system is not always merely random. Using the various properties of chaos, we have presented new possibility to estimate black hole's spin and magnetic field.

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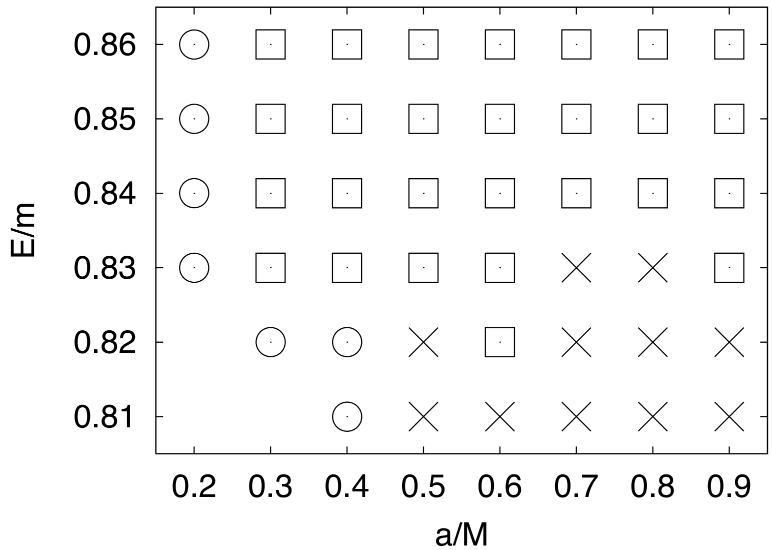

Figure 1: The phase diagram for the chaotic orbits. The patterns of the power spectrum for the chaotic orbits at grid points in two-dimensional $(a/M, E/m)$ configuration are tested. Here we set $Q = -30$ and $L/M = -7$. At the points where the symbols (◯) are marked, the $1/f$-type power spectra are observed. At the points where the symbols (□) are marked, the white-noise power spectra are observed. At the points where the symbols ($\times$) are marked, the orbits are regular (not chaotic).

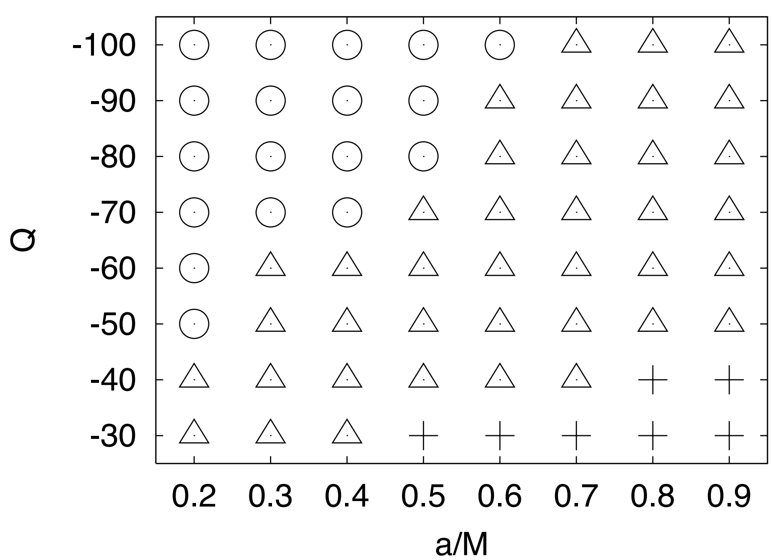

Figure 2: The phase diagram for the chaotic orbits. The patterns of the power spectrum for the chaotic orbits at grid points in two-dimensional $(a/M, Q)$ configuration are tested. Here we set $L/M = -7$. At the points where the symbols (◯) are marked, the $1/f$-type power spectrum is observed. At the points where the symbols (Δ) are marked, the $1/f$-type power spectra are observed for low energy, and the white-noise power spectra are observed for high energy. At the points where the symbols ($+$) are marked, the orbits are regular (not chaotic) for low energy, and the white-noise power spectra are observed for high energy.