PARTICLES AND BOUND STATES AND PROGRESS TOWARD UNITARITY AND SCALING

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Abstract

We present a survey of recent developments in constructive quantum field theory.

Introduction. The program of constructive quantum field theory starts with an approximate field theory whose existence is known and constructs a Lorentz covariant limit as the approximations are removed [50]. Frequently it has been convenient to work in the path space world of imaginary time [14,15,42]. The Osterwalder-Schrader axioms [34,35] give sufficient conditions on the Euclidean Green's functions (i.e. the path space theory in the case of bosons) to allow analytic continuation back to real (Minkowski) time and a verification of all Wightman axioms.

This program has been carried out in a number of models in space-time dimension $d<4$. Once a model has been constructed, the interesting questions involve its detailed properties and how these properties depend on the parameters. For space-time dimension $d=2$, considerable insight has been obtained into several models including the Sine-Gordon equation [11]. For $d=3$, recent work yields the first nontrivial model [9]. We describe some of these recent results below, but first we propose a program for $d=4$.

Consider a lattice ϕ_4^4 model, with lattice spacing ε. Using correlation inequalities of Lebowitz [18,30,47] or the Lee Yang theorem [32,33], we can bound the n-point Schwinger functions

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$$S^{(n)}(x_1,\ldots,x_n) = \int \phi(x_1)\ldots\phi(x_n)dq ,$$

defined as moments of the measure

$$dq = \frac{e^{-V(\phi)}\prod_x d\phi(x)}{\int e^{-V(\phi)}\prod_x d\phi(x)} ,$$

$$V(\phi) = \sum_x [\tfrac{1}{2}:(\nabla\phi)^2 + m_0^2\phi^2: + \lambda:\phi^4(x):] .$$

In fact $S^{(n)}$ is bounded by a sum of products of two point functions. Thus a sufficient condition for the existence of a ϕ^4 field theory (by the method of compactness and subsequences) is a bound, uniform in the lattice spacing ε, on the lattice approximation two point function. In order to prevent the identical vanishing of all n-point functions in the limit $\varepsilon \to 0$, we hold the physical mass fixed, and since $d = 4$, we perform a field strength renormalization.

Field strength renormalization assumes the existence of a one particle pole in the Fourier transform of the two point function, of strength $Z = Z(\varepsilon) \neq 0$. We define

$$S^{(n)}_{\text{ren}} = Z^{-n/2}S^{(n)} ,$$

and as above $S^{(n)}_{\text{ren}}$ is bounded in terms of $S^{(2)}_{\text{ren}} = Z^{-1}S^{(2)}$. Thus the essential missing steps are (i) existence of the one particle pole, so that Z is defined and not zero, for $\varepsilon > 0$, and (ii) control over $Z(\varepsilon)$ as $\varepsilon \to 0$. Since one expects $Z(\varepsilon) \to 0$ (infinite field strength renormalization), it is necessary to show that the rest of the mass spectrum in $S^{(2)}$ has a spectral weight converging to zero (in each bounded mass interval) with ε, so that $S^{(2)}_{\text{ren}}$ remains bounded.

The coupling constant renormalization, $\lambda = \lambda(\varepsilon) \to \infty$ as $\varepsilon \to 0$, should not be required for existence of the $\varepsilon \to 0$ limit, but if it is not performed, the limit should be a free field, for $d = 4$. See [24] for further discussion.

In the case $d = 2,3$, we can apply these same ideas to construct the "scaling limit" of (superrenormalizable) ϕ^4 models. In this case we do not require a cutoff for the approximate theories, but vary the bare parameters so $Z \to 0$ and $\lambda \to \infty$, while the physical mass remains fixed and the dimensionless charge approaches its critical value (characterized by the onset of symmetry breaking). We now consider the case $d = 2$ in more detail.

The scaling limit in ϕ_2^4 [24]. Consider an interaction Lagrangian

$$L^{\text{Int}} = -\frac{1}{2}\sigma\int :\phi^2:dx - \lambda\int :\phi^4:dx .$$

Here σ is related to the bare mass m_0, but does not equal m_0^2, since the free Lagrangian will contain a mass parameter also; see the appendix of [20] for a discussion. It is expected that a critical value $\sigma_c = \sigma_c(\lambda)$ of σ exists so that as $\sigma \downarrow \sigma_c$, the physical mass m goes to zero. The scaling limit combines the limit $\sigma \downarrow \sigma_c$ with an infinite scale transformation,

$$\lambda \to s\lambda, \quad \sigma \to s\sigma, \quad m^2 \to sm^2,$$

so that m is held fixed. The correct choice of s is given by the conditions

$$\lambda/\sigma \to \lambda/\sigma_c(\lambda) = (\lambda/\sigma(\lambda))_c ,$$

and in this limit $\lambda \to \infty$, $\sigma_c(\lambda) \to \infty$. Note that $\lambda/\sigma_c(\lambda)$ is dimensionless, hence a pure number independent of λ. In this limit all unrenormalized correlation functions converge (after passage to a subsequence). In taking the scaling limit, we also perform a field strength renormalization, so the one particle pole has residue 1. We expect $Z(\sigma) \to 0$ in the scaling limit.

As before, control over the renormalized two point function is sufficient for the existence of the scaling limit, and again the missing steps are (i), (ii) above. Other sufficient conditions are (α) negativity of the six point vertex function $\Gamma^{(6)} \le 0$, or (β) better than one particle decay of the inverse propagator $-\Gamma^{(2)}(x)$ (which for $x \neq 0$ equals the proper self energy part $\Pi(x)$) or (γ) absence of level crossings and control over CDD zeros of the momentum space two point function [24], and also [23]. Thus (α)-(γ) are central open issues.

A numerical analysis of the ϕ_1^4 model (anharmonic oscillator) is consistent with the validity of parts of (γ) [28]. An explicit calculation shows that $\Gamma^{(6)} < 0$ in the one-dimensional Ising model [39]. A related correlation inequality $S_T^{(6)} \ge 0$ (positivity of the connected part of $S^{(6)}$) has been announced by Cartier; recently Percus [38] and Sylvester [47] have given proofs. Computer studies of the Ising model by Sylvester indicate furthermore that $(-1)^{n+1} S_T^{(2n)} \ge 0$, [48]. The question of correlation inequalities will be discussed at greater length in the lecture of Simon.

In addition to its relation to the problem of constructing ϕ_4^4, the scaling limit is important in the renormalization group approach to the study of critical exponents; see for example [37]. In this connection we note that a number of critical exponents have been bounded from below by their canonical (mean field) values [17]. Furthermore we prove a priori bounds on renormalized coupling constants [21].

The problems (i) and (ii) above arise also in the scaling limit of the d-dimensional Ising model, $2 \le d \le 4$. For I_2, asymptotic calculations using Toeplitz determinants indicate that the required bounds are in fact satisfied [31, 49]. The model I_1 is already scale invariant. A formal interchange of limits identifies the scaling limit of I_d with the infinitely scaled ϕ_d^4 model. This statement provides a basis for the idea that spin 1/2 Ising model critical exponents are independent of the details of the lattice structure and are equal to those defined by a ϕ^4 field theory. It also suggests that $\phi^6, \phi^8, \ldots$ tri and multi-critical points are associated with higher spin Ising model tri and multi-critical points.

Particles and Unitarity [25,26]. Let $M = (H^2 - \vec{P}^2)^{1/2}$ be the mass operator. For weakly coupled $P(\phi)_2$ models, the spectrum of M is completely known in the interval $[0, 2m-\varepsilon]$. At zero, M has a simple eigenvalue with eigenvector Ω, the vacuum state. At m, M has an isolated eigenvalue, and the corresponding eigenspace, the space of one particle states, carries an irreducible representation of the Poincaré group. M has no other spectrum below $2m-\varepsilon$; here $\varepsilon \to 0$ and $m \to m_0$ (the bare mass) as the coupling tends to zero. The existence of an isometric S matrix and n-particle in and out states then follows from the Haag-Ruelle scattering theorem.

These results about particles (spectrum of M) are proved using a "cluster expansion," similar to the high temperature expansions in statistical mechanics [25,26]. These expansions, furthermore, give spectral information about the mass (generalized) eigenvectors on any bounded spectral interval, for λ sufficiently small. The main consequence of these expansions is the fact that the Euclidean correlation functions decay (become uncorrelated) as the points separate into clusters, and rate of decay is exponential in the separation distance. Let

$$X = x_1, \ldots, x_n, \qquad Y = y_1, \ldots, y_m$$

$$\Phi(X) = \prod_{i=1}^{n} \phi(x_i), \quad \Phi(Y) = \prod_{j=1}^{m} \phi(y_j).$$

A typical estimate is

$$\text{(1)} \qquad \int \Phi(X)\Phi(Y)d\phi - \int \Phi(X)d\phi \int \Phi(Y)d\phi = O(e^{-md})$$

where $m > 0$ is independent of X,Y and

$$d = \mathrm{dist}(X,Y)$$

is the separation distance. The subtraction in (1) can be recognized as a Euclidean vacuum subtraction. Let $\Omega_E \equiv 1$ be the Euclidean vacuum state and let $<\ ,\ >$ be the Euclidean inner product defined by the measure $d\phi$. Let P_0 be the orthogonal projection onto Ω_E. Then (1) can be written

$$\text{(2)} \qquad <\Phi(X),(I-P_0)\Phi(Y)> = O(e^{-md}) .$$

In fact, the exponential decay rate in (1),(2) is related to the spectrum of M. Define m_1 as the infimum over the exponential decay rates for different choices of X,Y. Then m_1 is the one particle mass (mass gap) in the spectrum of M.

We generalize (2) by replacing the projection $I-P_0$ with the projection onto the orthogonal complement of the subspace spanned by polynomials of degree $< n$. We then expect decay rates $m_n > m_1$. Let P_n denote the orthogonal projection onto the subspace spanned by Euclidean vectors

$$|X_n> \equiv (I - \sum_{i=0}^{n-1} P_i)\phi(x_1)\ldots\phi(x_n)\Omega .$$

The projection P_n can be written in terms of a kernel $P_n(X,Y)$, and in Dirac notation

$$\text{(3)} \qquad P_n = \int dXdY\,|X_n>P_n(X,Y)<Y_n| ,$$

where the integration extends over $R^{nd} \times R^{nd}$ and where the kernel $P_n(X,Y)$ is the inverse (in the sense of integral operators) to the kernel $<X_n|Y_n>$.

The kernel

$$P_1(X,Y) = -K_1(X,Y) = -\Gamma^{(2)}(x-y)$$

is the familiar inverse propagator, while the kernel $P_2(X,Y)$ is related to the Bethe-Salpeter kernel $K_2(X,Y)$, namely

$$P_2(X,Y) = -K_2(X,Y) + \frac{1}{2} K_1 \otimes K_1 .$$

In terms of Feynman graphs, K_2 is the connected part of $-P_2$. More generally, we can define an n-body Bethe-Salpeter kernel $K_n(X,Y)$ as the connected part of $-P_n$. In fact, P_n can be written as a sum of tensor products of Bethe-Salpeter kernels, generalizing the expressions for P_1, P_2 above:

$$(4) \qquad P_n(X,Y) = \sum_{r=1}^{n} \sum_{|\alpha_1|+\ldots+|\alpha_r|=n} (-1)^r \binom{n}{\alpha}^{-2} K_{|\alpha_1|} \otimes \ldots \otimes K_{|\alpha_r|} ,$$

where $\binom{n}{\alpha} \equiv n! \left(\prod_{i=1}^{r} |\alpha_i|! \right)^{-1}$ and the sum extends over all cluster decompositions into elastic channels. For instance, the term with $r=1$ is just $-K_n$, while the term with $r=n$ equals $(-1)^n (n!)^{-1} \Gamma \otimes \ldots \otimes \Gamma$.

The existence of the Bethe-Salpeter kernel K_n, for $n=1,2,3$ is proved in [19,45,23]. We will publish an analysis of (4). A much deeper question than existence of K_n is the magnitude of the exponential decay rates associated with these kernels. Let

$$|X_{n,r}\rangle \equiv (I - \sum_{i=0}^{n-1} P_i) \phi(x_1) \ldots \phi(x_r) \Omega .$$

We seek estimates generalizing (2) of the form

$$(5) \qquad \langle X_{n,r} | Y_{n,r} \rangle = O(e^{-m_n d}),$$

i.e. Euclidean cluster properties, or decay estimates

$$(6) \qquad K_n(X,Y) = O(e^{-\bar{m}_n d}) .$$

In order to analyze the decay of K_n, Spencer has derived a new cluster expansion [45] which generalizes [26] by giving higher particle subtractions. Explicitly in the case $n=2$, for weak coupling even $\mathcal{P}(\phi)_2$ models, Spencer proves that $\bar{m}_2 \geq 4m(1-\varepsilon)$, where $\varepsilon \to 0$ as the coupling tends to zero. He and Zirilli expect that the decay rate for the two body Bethe-Salpeter kernel will provide information related to asymptotic completeness, up to the threshold $\bar{m}_2$, i.e. for $M \leq 4m(1-\varepsilon)$ [46]. Similar methods give the decay rate for the part of $S_T^{(4)}$ that is two particle irreducible in each channel [2]; this amplitude is obtained by a second Legendre transformation.

If one imagines an extension of this structure analysis to arbitrary n, it appears that the allowed size of the weak coupling region would be n-dependent, and tend to zero as n tends to infinity. The question of dealing with the particle structure away from weak coupling is also of great importance. In this case we only have results for the ϕ^4 interaction, which is repulsive, as described in the following section.

Bound States. For $d=2,3$, bound states should occur for weak as well as strong attractive forces; they should be missing for repulsive forces. For single phase ϕ^4 models, it is known that no even bound states can occur [44,7,26]. We expect that odd bound states are also missing, cf [24]. For weak $(\phi^6-\phi^4)_2$ models, it is known that mass spectrum occurs in the bound state interval $(m,2m)$ [26]. The Bethe-Salpeter equation combined with improved decay estimates above should allow a complete analysis of the bound state problem for weak coupling $P(\phi)_2$ models; partial results have been obtained [46]. An interesting question is whether bound states occur in ϕ^4 models with symmetry.

Fermions and Many Body Systems. The original construction of the infinite volume limit for the $d=2$ Yukawa model was given by Schrader [41]; see also [14]. Some portions of this construction have been derived in a Euclidean formalism [43,1]. Aside from the increased simplicity which may accompany a covariant Euclidean construction, the Euclidean formalism is important as a natural framework for a cluster expansion and a study of particles.

Federbush [5,6] has simplified the Dyson-Lenard proof of the stability of matter. The methods were suggested in part by constructive field theory techniques, including techniques he previously employed for the Yukawa_2 model. [6] contains a cluster expansion, which should have a number of applications.

Phase Transitions. Fröhlich [10] has shown that the Euclidean decomposition of the measure $d\phi$ into time translation invariant components coincides with the direct integral decomposition of the associated quantum fields into pure phases.

Three Dimensions. The original semiboundedness proof for the ϕ_3^4 Hamiltonian [16] and related Schwinger function bounds [8,36] were given in a finite volume. A cluster expansion for ϕ_3^4 has been established by Feldman and Osterwalder [9], yielding the first nontrivial $d=3$ model of the Osterwalder-Schrader axioms. We can hope that further progress will soon bring ϕ_3^4 to the level of understanding we have for $d=2$.

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DISCUSSION

Masuo Suzuki (comment): I hope that by using your inequalities you can obtain such qualitative results as the dependence of critical exponents upon the dimensionality and potential-range parameter in your ϕ^4 model. In the ferromagnetic Ising model, I proved inequalities such as $\gamma(d) \geq \gamma(d+1) \geq \ldots$ and $\nu(d) \geq \nu(d+1) \geq \ldots$, using Griffiths' inequalities. (Physics Letters 38A (1972) 23.)