STATISTICAL PROPERTIES OF CURVATURE PERTURBATIONS GENERATED DURING INFLATION

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ABSTRACT

The stochastic dynamics of the scalar field responsible for inflation is considered in connection with the statistical properties of classical curvature perturbations which are generated by quantum fluctuations in that field. The combined effect of non–linearities in the scalar field and of perturbations in the metric makes curvature perturbations on large scales strongly non–Gaussian. The controversial issue of whether perturbations in our observable patch of the inflated universe are also non–Gaussian is discussed in terms of conditional probabilities.

INTRODUCTION

Density fluctuations generated during inflation (see, e.g., the review by A. Linde in these Proceedings) are usually considered to be Gaussian, as a general consequence of the required flatness of the potential for the *inflaton*, the scalar field which drives the accelerated universe expansion. Recently, however, it has been shown that both isocurvature [1)–4)] and curvature perturbations [5)–9)] can be characterized by non–Gaussian statistics. Due to the back–reaction of field fluctuations on the background geometry, phase correlations appear during the stochastic evolution of horizon–size inflaton modes, thus providing non–Gaussian initial conditions for the linear evolution of adiabatic perturbations. In fact, it has been shown [6)] that, for a wide class of potentials leading either to Linde's chaotic inflation [10)], or to power–law inflation [11),12)], fluctuations in the scalar field are non–Gaussian distributed around the classical trajectory: at the end of inflation the distribution for the gravitational potential fluctuations can be highly non–Gaussian [9)]. This is an important conclusion for theoretical cosmology since it opens the possibility of unexplored models for the formation of cosmic structures which, abandoning the random–phase paradigm, preserve the simplicity of the gravitational instability picture. Generally speaking one is faced with a new class of models which imply more structure on large scales than the standard cold dark matter model. Models of this type are presently under investigation in connection with their clustering properties on large scales [13)].

The mechanism for the generation of non–Gaussian adiabatic perturbations, which is discussed here is characterized by the absence of intrinsic lengths on cosmologically relevant scales, it therefore implies a *scale–invariant* fluctuation field (see, e.g., [14),15)]). Such a scale–invariance property is properly expressed by a simple scaling (up to negligible logarithmic corrections) of the peculiar gravitational potential $\Phi(\mathbf{x})$ in Fourier space at every time during matter dominance

$$\langle\Phi(\mu\mathbf{k}_1)\ldots\Phi(\mu\mathbf{k}_N)\rangle d^3(\mu k_1)\ldots d^3(\mu k_N) \approx \mu^{N(n_p-1)/2}\langle\Phi(\mathbf{k}_1)\ldots\Phi(\mathbf{k}_N)\rangle d^3k_1\ldots d^3k_N \quad (1)$$

where n_p is the primordial spectral index. For $n_p = 1$, Eq.(1) represents a generalization of the Zel'dovich criterion of scale–invariance to non–Gaussian fluctuations. This immediately translates into a statement on the time evolution of the N–point correlation functions $\xi^{(N)}(\mathbf{x}_1,\ldots,\mathbf{x}_N;t)$ for mass fluctuations, consistently defined in terms of the Zel'dovich approximation [16)], up to the time of first shell–crossing

$$\xi^{(N)}(\mathbf{x}_1,\ldots,\mathbf{x}_N;t_0) \approx \xi^{(N)}(\mu\mathbf{x}_1,\ldots,\mu\mathbf{x}_N;t) \quad (2)$$

provided that $\mu = [b(t)/b(t_0)]^{2/(n_p+3)}$, with b the growing mode of linear perturbations, proportional to $t^{2/3}$ in the matter dominated era and in a flat universe. This last property, however, only applies over suitably large scales where the curvature of the primordial spectrum introduced during the linear evolution of perturbations inside the horizon is unimportant. Although the potential Φ is simply related to the linear density fluctuation $\delta\varrho$ through the

Poisson equation

$$\triangle^{(3)}\Phi(\mathbf{x},t) = -4\pi G a^2(t)\delta\varrho(\mathbf{x},t), \tag{3}$$

with $a(t)$ the scale–factor, we prefer to define the mass density through the Zel'dovich approximation; this allows to extend the treatment to the mildly non–linear evolution and to take into account the constraint $\varrho(\mathbf{x}) \geq 0$. Simple toy–model examples of scale–invariant statistics are: the model proposed by [17], where the density fluctuation field is the convolution of two independent scale–free Gaussian processes; a model where the density field is the square of a Gaussian random process [as recently found by Bardeen (unpublished) in the analysis of a two–scalar field model for inflation]. As we shall see in the following, the non–Gaussian scale–invariant density fluctuations obtained from inflation are all of multiplicative type (see also [7]); this is an important property since a number of interesting phenomena are likely to occur in this case. Among these there is an interesting phenomenon called *intermittency* (see, e.g., [18]), which is clearly exhibited for instance by log–normal random fields; roughly speaking, intermittency consists in the occurrence, in realizations of the random field, of sporadic high spots where most of the intensity is stored, separated by large regions of reduced intensity.

A quite different mechanism for producing non–Gaussian and non–scale–invariant perturbations during inflation rests on the use of multiple (interacting) scalar fields during inflation as considered in refs. [4],[7]; the stochastic method has been recently extended to the multiple scalar field case (see, e.g., [19],[20]). In this case one can easily obtain non–Gaussian and/or non–scale–invariant perturbations both of adiabatic and isocurvature type (see, e.g., the contribution by S. Mollerach to these Proceedings).

STOCHASTIC INFLATION

It has been shown by many authors that the dynamics of the inflaton on scales larger than the comoving Hubble radius $r_H(t) \approx 1/\dot{a}(t)$ is accurately described by a stochastic approach (see, e.g., [21],[22]). One defines a *coarse–grained* variable $\varphi_{\mathbf{x}}(t)$ which is the average of the quantum scalar field over a volume of size $\sim r_H^3(t)$. In the spirit of the chaotic inflation scenario initial conditions are introduced by assuming that there are initial domains of size $\sim r_H^3(t_\star)$ characterized by a nearly homogeneous value $\varphi_\star$ for the scalar field. Provided that $\varphi_\star$ is large enough, a few Hubble times suffice to depress both the kinetic energy and the spatial gradients compared to the potential energy $V(\varphi)$. The resulting coarse–grained dynamics is *friction* dominated and can be described by a Langevin–type equation

$$\dot{\varphi}_{\mathbf{x}} = -\frac{1}{3H(\varphi_{\mathbf{x}})}\frac{\partial V(\varphi_{\mathbf{x}})}{\partial \varphi_{\mathbf{x}}} + \frac{H^{3/2}(\varphi_{\mathbf{x}})}{2\pi}\eta_{\mathbf{x}}(t), \tag{4}$$

where $H(\varphi_{\mathbf{x}}) \approx \sqrt{V(\varphi_{\mathbf{x}})/3\sigma^2}$ and $\sigma \equiv 1/\sqrt{8\pi G}$. In this approach $\mathbf{x}$ labels the coarse–grained variable in different cells. In writing Eq.(4) one implicitly assumes a perturbed Friedmann line–element in an appropriate synchronous gauge (see, e.g., [23])

$$ds^2 \approx dt^2 - a^2(\mathbf{x},t)dl^2, \tag{5}$$

with a local scale–factor

$$a(\mathbf{x},t) \approx a(\mathbf{x},t_\star)\exp\int_{t_\star}^{t} H(\varphi_{\mathbf{x}})dt'. \tag{6}$$

Non–diagonal scalar perturbations of the metric, which are also initially present in this gauge, are quickly depressed, on scales larger than the horizon, by the inflationary expansion. The first term on the r.h.s. of Eq.(4) plays the role of a classical convective force, while the second one represents the diffusion process induced by fine–grained quantum fluctuations. The dependence of H on φ makes the stochastic process a multiplicative one. The noise $\eta_{\mathbf{x}}(t)$, which has zero mean, is accurately approximated by a stationary Gaussian process with auto–correlation function

$$\langle \eta_{\mathbf{x}}(t)\eta_{\mathbf{x}'}(t')\rangle = \hbar\ j_0\big(|\mathbf{x}-\mathbf{x}'|/r_H(t)\big)\ \delta(t-t'), \tag{7}$$

where j_0 is the zero order spherical Bessel function. Because of the *white–noise* character of η, with respect to its time dependence, $\varphi(t)$ is *Markovian.* The quantity $\mathcal{P}(\varphi,t)\ d\varphi = \langle \delta\big(\varphi - \varphi[\eta_{\mathbf{x}}(t)]\big)\rangle_\eta\ d\varphi$, yielding the (*one–particle*) probability that, in a randomly chosen point $\mathbf{x}$, $\varphi_{\mathbf{x}}$ takes a value in the interval $\varphi, \varphi + d\ \varphi$, evolves according to a Fokker–Planck equation (in the Einstein–Smoluchowski limit)

$$\frac{\partial \mathcal{P}}{\partial t} = \frac{\partial}{\partial\varphi}\left[\left(\frac{2\sigma}{3^{1/2}}\frac{\partial V^{1/2}}{\partial\varphi}\right)\mathcal{P} + \frac{\hbar}{4\pi^2}\ \frac{V^{3/4}}{3^{3/2}\sigma^3}\frac{\partial}{\partial\varphi}\left(V^{3/4}\mathcal{P}\right)\right]. \tag{8}$$

Once the inflaton potential $V(\varphi)$ has been specified, one looks for a time–dependent solution with the initial condition $\mathcal{P}(\varphi,t_\star) = \delta(\varphi-\varphi_\star)$ corresponding to a homogeneous configuration $\varphi_\star$, with potential energy $V_\star \approx 3\sigma^2 H_\star^2$ (one needs $H_\star/\sigma \lesssim 8\pi/\sqrt{3}$, in order not to exceed the Planck energy).

It should be recalled that the solution of Eq.(8) with a delta–like initial condition is actually a conditional probability (also called *transition probability* in this context): it gives the probability that our stochastic process takes the value φ at time t (in a randomly selected position) given that the result of a measurement was $\varphi_\star$ at time $t_\star$ (Smoluchowski called it *probability after effects*). Because of the Markovian character of the process the sharp condition at $t_\star$ cancels any memory of the evolution preceding $t_\star$; also, for suitably short time intervals after $t_\star$ the form of the probability is strongly dominated by the constraint, while for times long enough the initial distribution is essentially forgotten.

The Langevin equation (4), together with the η correlation function Eq.(7), actually contains much more information than the (one–particle) probability $\mathcal{P}$, for it takes into account the space–correlation properties of the distribution, that is, it allows to obtain the whole probability density functional. The probability density functional evolves according to a suitable *functional Fokker–Planck equation* [such a complete treatment has been sketched, for instance, by Rey [24]), although he completely disregarded the role of metric perturbations]. Another effect which should be taken into account when dealing with the space–distribution of the coarse–grained field is connected with the different weights to be assigned to different cells due to fluctuations in the coarse–graining volume. This problem is dealt with by many authors by multiplying the probability by the volume factor $a^3(\mathbf{x},t)$, obtained from Eq.(6). This correction is at the origin of the so-called *eternally existing self-reproducing inflationary universe* (see, e.g., [23),25)]); for our purposes neglecting such a correction is a minor approximation since this is expected to mainly affect the global properties of the universe and the evolution of the probability distribution at early times, when the local fluctuations of the Hubble constant can be very large.

It has been shown [6)] that the dynamics described by Eq.(8) [or Eq.(4)] presents universal properties for a large class of models leading either to chaotic or to power–law inflation. These can be summarized as follows. At early times the diffusion term in Eq.(8) causes the initial delta function to spread around its maximum which starts moving, due to the convective term. Convection, at this stage, can be approximated by a constant positive force (namely $-V'/3H$ evaluated at $\varphi_\star$), later on, however, force gradients start to be felt: *Brownian particles* which are closer to the minimum of the potential suffer smaller attraction compared to more distant ones; at the same time the diffusion coefficient becomes smaller and smaller as the minimum is approached and can be safely neglected. This causes a shrinking of the distribution at late times characterizing the so called *scaling regime* (see, e.g., [26)]): at the *scaling time* t_{sc} the system undergoes a transition from a diffusion dominated regime (due to quantum fluctuations) to a convection dominated regime (due to classical non–linearities), or from a *disordered phase* to a *macroscopically ordered phase*. In this picture inflation is described as the non–equilibrium decay of the system from the unstable state $\varphi_\star$ to the minimum of the potential, with the whole coarse–grained field distribution undergoing slow–rolling down. The peaking of the probability at the onset of the scaling regime was noticed by a number of authors [24),27),28)]. During the scaling regime the distribution is strongly peaked around the classical homogeneous configuration $\varphi_{cl}(t)$ [(the solution of Eq.(4) when η is set to zero, or when $\hbar \to 0$], with fluctuations giving rise to classical curvature perturbations. In this *scaling limit* the solution tends to a self–similar time–independent function $\mathcal{W}$ of a single *scaling variable* ξ: $\mathcal{P}(\varphi,t)\, d\varphi \sim \mathcal{W}(\xi)\, d\xi$. Such a scaling variable can be chosen so that it is, with very good accuracy (or even exactly), Gaussian with zero mean and two–point function

$$\langle \xi_{\mathbf{x}}(t)\xi_{\mathbf{x}'}(t')\rangle = \frac{1}{2\pi^2}\int_{1/r_H(t_\star)}^{1/r_H(t_{min})} dk k^2 (P_0/k^3) j_0(k|\mathbf{x}-\mathbf{x}'|), \tag{9}$$

where $t_{min} = min(t,t')$. Only wavelengths that left the horizon during the interval $t_\star \div t_{min}$ contribute to the integral in Eq.(9). The ξ power–spectrum has a *flicker–noise* form $P(k) = P_0 k^{-3}$. The amplitude P_0 (which may possibly contain a residual logarithmic k–dependence) as well as the non–linear transformation between φ and ξ depend upon the precise form of the inflaton potential.

INFLATIONARY MODELS

We shall restrict our discussion to two simple models: chaotic inflation driven by a quartic inflaton potential [10)] and power–law inflation based on an exponential potential, as first proposed in ref. [12]. Chaotic inflation can be driven by a quartic potential $V(\varphi) = (\lambda/4)\varphi^4$. The classical solution of the φ dynamics in the slow–rolling down phase is described by $\varphi_{cl}(t) = \varphi_\star \exp(-2\sqrt{\lambda/3}\ \sigma\Delta t)$, where $\Delta t = t - t_\star$. If the expansion is dominated by the classical homogeneous mode, the scale–factor is given by

$$a(t) = a_\star \exp\{(H_\star/4\sigma)(3/\lambda)^{1/2}\left[1 - \exp(-4\sqrt{\lambda/3}\ \sigma\Delta t)\right]\}. \tag{10}$$

Inflation is expected to end when the kinetic energy equals the potential energy, i.e. when $\varphi = \pm 2\sqrt{2}\sigma$. The exponential potential $V(\varphi) = V_\star \exp[-\lambda(\varphi - \varphi_\star)/\sigma]$, leads to power–law inflation

$$a(t) = a_\star\left[1 + \lambda^2 H_\star \Delta t/2\right]^{2/\lambda^2}, \tag{11}$$

for any $\lambda < \sqrt{2}$. Unless the potential is suitably modified to allow for reheating to occur, inflation lasts forever, the ratio of kinetic and potential energy being constant along the classical trajectory $\varphi_{cl}(t) = \varphi_\star + (2\sigma/\lambda)\ln(1 + \lambda^2 H_\star \Delta t/2)$.

For the quartic potential one can solve exactly the Langevin equation and find, for $\delta\varphi_{\mathbf{x}}(t) = \varphi_{\mathbf{x}}(t) - \varphi_{cl}(t)$ [$\varphi_{cl}(t)$ generally differs from $\langle\varphi(t)\rangle$ by a small, time–dependent, space–homogeneous quantity which does not affect observable quantities]

$$\delta\varphi_{\mathbf{x}}(t) = \pm\ \sigma \left(\frac{12H_\star^2}{\lambda\sigma^2}\right)^{1/4} \exp[-2\sqrt{\lambda/3}\ \sigma\Delta t]\left[|1 + \xi_{\mathbf{x}}(t)|^{-1/2} - 1\right], \tag{12}$$

where the solution with the plus (minus) sign must be taken if $\varphi_\star$ is positive (negative). In this case $P_0 \approx \hbar(\lambda/3)^{1/2}(H_\star/\sigma)[1 - 4(\lambda/3)^{1/2}(\sigma/H_\star)\ln(k/a_\star H_\star)]$. For the exponential potential appropriate use of the *scaling approximation* yields, for large times

$$\delta\varphi_{\mathbf{x}}(t) = \frac{4\sigma}{\lambda^3 H_\star t}\left[|1 + \xi_{\mathbf{x}}(t)|^{2/3} - 1\right] \tag{13}$$

and $P_0 = (9\hbar/16)[\lambda^2/(2 - \lambda^2)](H_\star/\sigma)^2$. The random–phase approximation in this context would amount to expand the r.h.s. of Eqs.(12) and (13) to first order in ξ; however ξ can have a large r.m.s. value, depending on the value of $H_\star$, so that such an approximation would fail in the general case. In the exponential potential model, for instance,

$$\xi_{rms}(t) = \left(\frac{\hbar}{2\pi^2}\right)^{1/2} \frac{3H_\star}{4\sigma} \ln^{1/2}(1 + \lambda^2 H_\star \Delta t/2), \tag{14}$$

which, for times after the onset of the scaling regime $\Delta t_{sc} \approx 2(\sqrt{e} - 1)\lambda^{-2}H_\star^{-1}$, can be larger than unity. This fact is at the origin of the non–Gaussian behaviour of $\delta\varphi$ discussed by Matarrese, Ortolan and Lucchin [6]. For the quartic potential model, for times much larger than $\Delta t_{sc} \approx (3/\lambda)^{1/2}(1/8\sigma)\ln 3$, due the the large value of $\xi_{rms}(t)$, one can approximately write $\varphi_{\mathbf{x}}(t) \propto |\xi_{\mathbf{x}}(t)|^{-1/2}$. Positive moments of φ, being related to negative moments of a semi–Gaussian distribution (i.e., the distribution for $|\xi|$), are infinite (as for the Cauchy distribution), except for the first one, $\langle\varphi\rangle$. One finds that the probability of crossing a level ν

times a suitable *effective dispersion* $\langle \delta\varphi_{eff}^2 \rangle^{1/2}$ is far from the Gaussian expectation already for small ν. This result holds, in the scaling regime, for suitably large values of $H_\star/\sigma$, independently of the value of λ and of time [in disagreement with Hodges [29], who incorrectly used a random–phase approximation to estimate these crossing probabilities], as follows from the scale–invariance property

$$\mathcal{P}(\varphi, t) \approx \mu \mathcal{P}(\mu\varphi, t - \sqrt{3/4\lambda\sigma^2}\ln\mu). \tag{15}$$

For the exponential potential one finds at large times $\varphi_{\mathbf{x}}(t) \propto |\xi_{\mathbf{x}}(t)|^{2/3}$ and

$$\langle \varphi^N \rangle \sim \pi^{-1/2}\Gamma\left(\frac{1}{2} + \frac{N}{3}\right)\left(\frac{2^{7/3}\sigma}{\lambda^3 H_\star t}\right)^N \xi_{rms}^{2N/3}(t), \tag{16}$$

with ξ_{rms} given in Eq.(14). In the scaling regime the N–th connected moments normalized to the power $N/2$ of the variance relax to constant non–zero values: clear evidence for non–Gaussian scale–invariant behaviour of the statistics. These dimensionless ratios do not depend neither on the value of λ, nor on time (scale), nor on the initial condition through $H_\star$. The role of $H_\star$, however, is more subtle: it must be large enough so that the system has well entered the scaling regime when the cosmologically interesting scales leave the inflationary horizon. It is clear that a low value of $H_\star$ (just enough to solve the horizon problem), as that quoted by Kofman *et al.* [7], keeps the system in the diffusive regime, where the fluctuations are practically Gaussian, and implies that our universe is exactly homogeneous on scales larger than the Hubble radius. This property is much stronger than what required by the isotropy of the cosmic microwave background, which only demands (for reasonable fluctuation spectra) that the r.m.s. density fluctuation on the scale of the present horizon is less than about 10^{-5}. Fluctuations on super–horizon scales only determine the local value of *average quantities.*

CURVATURE FLUCTUATIONS

For scale–free inflaton potentials such as those presented in the previous section these results can be expressed in an a simple form as follows: the scaling approximation amounts to replacing the Langevin equation (4) by the following effective equation

$$\dot\varphi_{\mathbf{x}}(t) = F_{cl}(t)\ \varphi_{\mathbf{x}}(t) + c\ \varphi_{\mathbf{x}}^{\beta}(t)\ \eta_{\mathbf{x}}(t), \tag{17}$$

where $F_{cl}(t)$ is some function of time, c is a constant and the power β depends upon the inflaton potential. This equation is exactly solved by

$$\varphi_{\mathbf{x}}(t) = \varphi_{cl}(t)|1 + \xi_{\mathbf{x}}(t)|^{1/(1-\beta)}, \tag{18}$$

where $\varphi_{cl}(t)$ is determined by $F_{cl}(t)$ through

$$\varphi_{cl}(t) = \varphi_\star \exp \int_{t_\star}^{t} dt' F_{cl}(t') \tag{19}$$

[except for the exponential potential case, where the situation is more involved as shown by Eq.(13), Eq.(19) is actually the definition of $F_{cl}(t)$] and

$$\xi_{\mathbf{x}}(t) = (1-\beta)\; c \int_{t_\star}^{t} dt' \varphi_{cl}^{\beta-1}(t')\; \eta_{\mathbf{x}}(t'). \tag{20}$$

In this simplified treatment the quartic potential, for which this approach is exact, corresponds to $\beta = 3$; more in general, inflaton potentials of the type $V \propto \varphi^{2n}$ are described by equation (17) with $\beta = 3n/2$ and the exponential potential, at late times, corresponds to $\beta = -1/2$. Note that for $\beta = 1$ the solution of this equation yields a log–normal process, while if β goes to zero in the equation (in the absence of boundary conditions) one gets a Gaussian process with $\delta\varphi_{\mathbf{x}}(t)/\varphi_{cl}(t) = \xi_{\mathbf{x}}(t)$. Also the model considered by Bardeen which is derived for two interacting scalar fields is formally described by this equation for $\beta = 1/2$. Therefore Eq.(17) describes a whole set of models, parametrised by β, which goes from the Gaussian case $\beta = 0$ to the extreme case of multiplicative stochastic process $\beta = 1$, leading to the intermittency phenomenon: the log–normal process.

The coarse–grained field fluctuation $\delta\varphi_{\mathbf{x}}(t)$ contains all wavelengths larger than the horizon size at the time t. In order to obtain the peculiar gravitational potential $\Phi(\mathbf{x})$ it is necessary to Fourier transform $\delta\varphi$ at each time t keeping only that mode $k \approx 1/r_H(t)$ that crosses the Hubble radius then. We can obtain the Fourier modes of the gravitational potential fluctuation field which reentered the horizon during the matter dominated era, by using their approximate constancy (the different behaviour of fluctuations which entered the horizon during radiation dominance is properly accounted for by the transfer function). One has

$$\Phi(\mathbf{k}) \approx -\frac{3T(k)}{5}\, \frac{H_{cl}(t_1)}{\dot\varphi_{cl}(t_1)}\, \delta\varphi(\mathbf{k}, t_1), \tag{21}$$

where t_1 is the time when the wave–number k left the horizon during inflation and $T(k)$ is the transfer function appropriate for the type of scenario one is considering. One will therefore find

$$\Phi(\mathbf{k}) \approx \frac{3T(k)}{10}\left(\frac{3H_\star^2}{\lambda\sigma^2}\right)^{1/2} \exp(-4\sqrt{\lambda/3}\;\sigma\; t_1) \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\big[|1+\xi_{\mathbf{x}}(t_1)|^{-1/2} - 1\big], \tag{22}$$

for the quartic potential and

$$\Phi(\mathbf{k}) \approx -\frac{12T(k)}{5\lambda^4 H_\star t_1} \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\big[|1+\xi_{\mathbf{x}}(t_1)|^{2/3} - 1\big], \tag{23}$$

for the exponential potential, at times much larger than the scaling time. It is clear that the non–linear transformation from the Gaussian variable ξ to $\delta\varphi$ implies that all Fourier modes of ξ contribute to $\Phi(\mathbf{k})$. As we have shown before, in the scaling regime one can approximately write $\big[|1+\xi_{\mathbf{x}}(t)|^{\alpha} - 1\big] \sim |\xi_{\mathbf{x}}(t)|^{\alpha}$ in the r.h.s. of Eqs.(22) and (23) where $\alpha = 1/(1-\beta)$ is respectively $-1/2$ and $2/3$. Note that this is the opposite of the random–phase approximation, valid in the initial diffusion dominated regime, where one writes $\big[|1+\xi_{\mathbf{x}}(t)|^{\alpha} - 1\big] \sim \alpha\; \xi_{\mathbf{x}}(t)$.

Of course, if one had linearized from the beginning the evolution equation for $\delta\varphi$, the coarse-grained fluctuations would come out proportional to ξ and then Gaussian (non-Gaussian fluctuations can be obtained in the linear approximation by interacting multiple scalar fields, as in ref. [4]): the linear approximation for $\delta\varphi$, therefore, does not adequately follow the dynamics during the convective regime. The non-linear transformation $\xi \rightarrow \Phi(\mathbf{x})$ implies that fluctuations of ξ on all scales, even of super-horizon size (where, due to the k^{-3} tail of the spectrum, ξ largely fluctuates) affect the present statistics of sub-horizon density perturbations which will then be non-Gaussian. The statistics of the peculiar gravitational potential so obtained can thus be used for building up the initial particle distribution in space and velocity through the Zel'dovich algorithm; this can then be used for evolving an N-body code [13]. Note that the Φ power-spectrum $P_\Phi(k)$ is simply related to that for the underlying Gaussian process ξ. In particular for the exponential potential model

$$P_\Phi(k) \propto (k/k_\star)^{-2\lambda^2/(2-\lambda^2)} k^{-3} \ln^{-1/3}(k/k_\star), \tag{24}$$

with $k_\star = a_\star H_\star$. The logarithmic correction to the power-law spectral shape is the only effect of the t_1 dependence of $\xi_{\mathbf{x}}(t_1)$ in the previous expressions; this is a general result which allows to treat ξ as being essentially time-independent in expressions such as (22) and (23). We can therefore conclude that inflationary models of the type considered here give curvature perturbations approximately described by the following form for the gravitational potential perturbation smoothed over the scale R

$$\Phi_R(\mathbf{x}) = \int d^3y f_R(\mathbf{y}-\mathbf{x}) |\xi(\mathbf{y})|^\alpha + const, \tag{25}$$

where $\xi(\mathbf{x})$ is now simplified to a Gaussian process with power-spectrum $P(k) = P_0 k^{-3}$, for $k_\star \leq k \leq k_{max}$, and $P(k) = 0$ elsewhere, k_{max} being a wave-number much larger than any mode of cosmological interest. The function f_R has Fourier transform

$$f_R(k) \propto W_R(k) T(k) k^{(n_p-1)/2}, \tag{26}$$

where $W_R(k)$ is a suitable low-pass filter which cuts off scales much smaller than R. Equations (25) and (26) allow to perform simulations of the non-Gaussian models described here, following standard methods (see [13]). It is important to note that the arbitrary additive constant in Eq.(25) has no observable effects.

Some authors have remarked (see, e.g., [7]) that the probability distribution for $\delta\varphi$ derived from the Fokker-Planck equation (8) cannot be used directly to yield density fluctuations for our observable patch of the inflated universe. As we have discussed here (see also [30],[9]) the obtaining of the actual density fluctuation field actually requires a more complete treatment taking into account the spatial variation of the quantum noise. This can be done by resorting to a functional Fokker-Planck equation or by using the Langevin equation (4) together with the η correlation function (7), as discussed so far. These authors have also argued that one should somehow consider a different type of initial condition for Eq.(8), or a conditional probability that the coarse-grained field has a certain value φ at time t, i.e., on the scale R leaving the horizon at that time, given that it a had a particular value φ_0 at the time t_0, corresponding to the present horizon scale R_0. The time t_0 would typically correspond to about 60 e–foldings before the end of inflation; this is practically equivalent to starting to evolve the probability at t_0 from a delta function centered on a value $\varphi_0 << \varphi_\star$. The underlying idea is that one should somehow constrain the density field to be homogeneous on the horizon scale, since we cannot perceive inhomogeneities on scales much larger than R_0. It is clear that this condition prevents the system from entering the scaling regime, because one has too little time. Fluctuations in the gravitational potential, however, still happen to be small, because the dispersion had little time to grow. In this case one can easily show that for most values of φ_0 the fluctuations are essentially Gaussian, with the exception of very high fluctuations, rare events which do not perceive the constraint. Also, for a few values of the constraint φ_0, the conditional probability is highly non–Gaussian and similar to the unconstrained one: these rare values of φ_0 give in fact the dominant weight in the unconstrained probability, as a consequence of the intermittency property, referred above. We believe, however, that the constraint imposed by the value of the coarse–grained field φ_0 is unphysical, because it ignores the arbitrary additive constant which enters the definition of the gravitational potential. Lucchin, Matarrese and Ortolan [9] therefore consider a different quantity, which is not affected by this indeterminacy: the gradient of the gravitational potential $\mathbf{\Psi}$, on a given scale R, this being proportional to the peculiar velocity field. For simplicity we consider the quantity $\Delta\Psi \equiv [\mathbf{\Psi}(\mathbf{x}) - \mathbf{\Psi}_0(\mathbf{x})] \cdot \mathbf{n}$, with $\mathbf{\Psi}_0$ the same quantity evaluated on the scale R_0. This quantity is proportional to the peculiar velocity, projected along the direction n, measured by an observer placed in $\mathbf{x}$ in the local rest frame set by the cosmic background radiation. Both the probability for $\Delta\Psi$ and the one for the same quantity conditioned by the value of $\Psi_0 \equiv \mathbf{\Psi}_0 \cdot \mathrm{n}$ are highly non–Gaussian, for essentially all values of the constraint. These non–Gaussian distributions are characterized by power–law tails (instead of exponential ones, like in the Gaussian case), namely

$$\mathcal{P}(\Delta\Psi) \sim |\Delta\Psi|^{-\gamma}, \qquad \mathcal{P}(\Delta\Psi|\Psi_0) \sim |\Delta\Psi|^{-(1+\gamma)}, \tag{27}$$

with $\gamma = 1 + 1/(1 - \alpha)$, which typically imply diverging moments. [More details will be given in the paper by Lucchin, Matarrese and Ortolan [9].] The physical consequence of this fact is that high peculiar velocities on large scales are much more likely than for a Gaussian field with the same power–spectrum. This qualitative conclusion is confirmed by numerical simulations of initially non–Gaussian distributions for the gravitational potential [13].

CONCLUSIONS

The stochastic approach allows to study the dynamics of inflation on scales larger than the Hubble radius. It accounts for the generation of large–scale classical fluctuations from quantum oscillations inside the horizon, the effect of non–linearities on the evolution of inflaton perturbations and the back–reaction of matter fluctuations on the background geometry.

If the spatial dependence of the fine-grained correlation function is kept, the whole spatial pattern of the scalar field fluctuations is known: from this the statistical distribution of the gravitational potential fluctuation field can be completely reconstructed. A generic feature of models leading to chaotic or to power-law inflation is the occurrence of a scaling regime where the coarse-grained distribution sharply peaks around the classical solution. In this regime the distribution is non-Gaussian and scale-invariant, due to the dominance of the inflaton non-linearities over the diffusion caused by quantum fluctuations. The deviation from the Gaussian behaviour does not depend on the strength λ of the non-linearity, on time (scale), and on the initial condition, provided the latter permits the system to enter the scaling regime. Because the probability is peaked, its bulk properties may be described by a Gaussian centered on $\varphi_{cl}(t)$ with suitable dispersion, but this approximation would fail in estimating the likelihood of rare events for which the actual distribution is required.

The scale-invariance of the inflaton is reproduced by the peculiar gravitational potential field, during its linear evolution. As a consequence, the N-point mass correlation functions will obey Eq.(2), with a spectral index n_p determined by the inflationary parameters; in the exponential potential model, for instance, $n_p = 1 - 2\lambda^2/(2-\lambda^2) \leq 1$ (up to negligible logarithmic corrections). The non-Gaussian scaling invariance and the multiplicative character of the primordial perturbation field will affect the properties of the universe on large scales, in particular the probability for the occurrence of rare events, such as high peculiar velocities, large empty regions, long filaments and *great attractors.* Moreover, one should expect interesting consequences for the statistics of the cosmic microwave background anisotropies.

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